Optimization-Based AMP for Phase Retrieval: The Impact of Initialization and ℓ_2 Regularization

Junjie Ma, Ji Xu, and Arian Maleki

***Abstract*— We consider an ℓ_2-regularized non-convex optimization problem for recovering signals from their noisy phaseless observations. We design and study the performance of a message passing algorithm that aims to solve this optimization problem. We consider the asymptotic setting $m, n \to \infty$, $m/n \to \delta$ and obtain sharp performance bounds, where m is the number of measurements and n is the signal dimension. We show that for complex signals, the algorithm can perform accurate recovery with only $m = ((64/\pi^2) - 4)n \approx 2.5n$ measurements. Also, we provide a sharp analysis on the sensitivity of the algorithm to noise. We highlight the following facts about our message passing algorithm: 1) adding ℓ_2 regularization to the non-convex loss function can be beneficial and 2) spectral initialization has a marginal impact on the performance of the algorithm. The sharp analyses, in this paper, not only enable us to compare the performance of our method with other phase recovery schemes but also shed light on designing better iterative algorithms for other non-convex optimization problems.**

***Index Terms*— Phase retrieval, Wirtinger flow, amplitude flow, approximate message passing, phase transition.**

I. Introduction

A. Informal Statement of Our Results

PHASE retrieval refers to the task of recovering a signal $\boldsymbol{x}_* \in \mathbb{C}^{n\times 1}$ from its m phaseless linear measurements:

$$y_a = \left|\sum_{i=1}^{n} A_{ai} x_{*,i}\right| + w_a, \quad a = 1, 2, \ldots, m, \qquad \text{(I.1)}$$

where $x_{*,i}$ is the ith component of $\boldsymbol{x}_*$, and $w_a \sim \mathcal{CN}(0, \sigma_w^2)$ a Gaussian noise. Throughout this paper, we assume that $A_{ai} \sim \mathcal{CN}(0, 1/m)$ and $\{A_{ai}\}$ are independent identically distributed (i.i.d). The recent surge of interest [1]–[23] has led to a better understanding of the theoretical aspects of this problem. Thanks to such research we now have access to several algorithms, inspired by different ideas, that are theoretically guaranteed to recover $\boldsymbol{x}_*$ exactly in the noiseless setting. Despite all this progress, there is still a gap between the theoretical understanding of the recovery algorithms and what practitioners would like to know. For instance, for many algorithms, including Wirtinger flow [4], [5] and amplitude flow [6], [7], the exact recovery is guaranteed with either $cn \log n$ or cn measurements, where c is often a fixed but large constant that does not depend on n. In both cases, it is often claimed that the large value of c or the existence of $\log n$ is an artifact of the proving technique and the algorithm is expected to work with cn for a reasonably small value of c. However, such claims do not address the following questions:

Q.1 Which algorithm should we use? Since the theoretical analyses are not sharp, they do not shed any light on the relative performance of different algorithms. Answering this question through simulations is very challenging too, since many factors including the distribution of the noise, the true signal $\boldsymbol{x}_*$, and the number of measurements may have impact on the answer.

Q.2 When can we trust the performance of these algorithms in the presence of noise? Suppose for a moment that we know the minimum number of measurements that is required for the exact recovery through simulations. Should we collect the same number of measurements in the noisy settings too?

Q.3 What is the impact of initialization schemes, such as spectral initialization? Can we trust these initialization schemes in the presence of noise? How should we compare different initialization schemes?

Researchers have developed certain intuition based on a combination of theoretical and empirical results, to give heuristic answers to these questions. However, as demonstrated in a series of papers in the context of compressed sensing, such heuristics are sometimes inaccurate [24]. To address Question Q.1, several researchers have adopted the asymptotic framework $m, n \to \infty$, $m/n \to \delta$, and provided sharp analyses for the performance of several algorithms [20]–[22]. This line of work studies recovery algorithms that are based on convex optimization. In this paper, we adopt the same asymptotic framework and study the following popular non-convex problem, known as amplitude-based optimization [7], [6]:

$$\min_{\boldsymbol{x}} \sum_{a=1}^{m} (y_a - |(\boldsymbol{A}\boldsymbol{x})_a|)^2 + \frac{\mu}{2}\|\boldsymbol{x}\|_2^2. \qquad \text{(I.2)}$$

where $(\boldsymbol{A}\boldsymbol{x})_a$ denotes the a-th entry of $\boldsymbol{A}\boldsymbol{x}$. Note that compared to the optimization problem discussed in [6] and [7], (I.2) has an extra ℓ_2-regularizer. Regularization is known to

Manuscript received January 8, 2018; revised August 31, 2018; accepted December 15, 2018. Date of publication January 18, 2019; date of current version May 20, 2019. This work was supported by the National Science Foundation under Grant CIF 1420328. This paper was presented at the 35th International Conference on Machine Learning.

J. Ma and A. Maleki are with the Department of Statistics, Columbia University, New York City, NY 10027 USA (e-mail: jm4520@columbia.edu; arian@stat.columbia.edu).

J. Xu is with the Department of Computer Science, Columbia University, New York City, NY 10027 USA (e-mail: jixu@cs.columbia.edu).

Communicated by A. H. Sayed, Associate Editor for Signal Processing.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2019.2893254

reduce the variance of an estimator and hence is expected to be useful when $\boldsymbol{w} \neq \mathbf{0}$. However, as we will try to clarify later in this section, since the loss function $\sum_{a=1}^{m} (y_a - |(\boldsymbol{Ax})_a|)^2$ is non-convex, regularization can help the iterative algorithm that aims to solve (I.2) even in the noiseless settings.

Since (I.2) is a non-convex problem, the algorithm to solve it matters. In this paper, we study a message passing algorithm that aims to solve (I.2). As a result of our studies we

1) present sharp characterization of the mean square error (even the constants are sharp) in both noiseless and noisy settings.
2) present a quantitative characterization of the gain initialization and regularization can offer to our algorithms.

Furthermore, the sharpness of our results enables us to present a quantitative and accurate comparison with convex optimization based recovery algorithms [20]–[22] and give partial answers to Question Q.1 mentioned above. Below we introduce our message passing algorithm and informally state some of our main results. The careful and accurate statements of our results are postponed to Section II.

Following the steps proposed in [25], we obtain the following algorithm, called *Approximate Message Passing for Amplitude-based optimization* (AMP.A). Starting from an initial estimate $\boldsymbol{x}^0 \in \mathbb{C}^{n\times 1}$, AMP.A proceeds as follows for $t \geq 0$:

$$\boldsymbol{p}^t = \boldsymbol{Ax}^t - \frac{\lambda_{t-1}}{\delta} \cdot \frac{g(\boldsymbol{p}^{t-1}, \boldsymbol{y})}{-\mathrm{div}_p(g_{t-1})},$$
$$\boldsymbol{x}^{t+1} = \lambda_t \cdot \left(\boldsymbol{x}^t + \boldsymbol{A}^{\mathrm{H}} \frac{g(\boldsymbol{p}^t, \boldsymbol{y})}{-\mathrm{div}_p(g_t)} \right).$$

In these iterations

$$g(p, y) = y \cdot \frac{p}{|p|} - p,$$

and

$$\lambda_t = \frac{-\mathrm{div}_p(g_t)}{-\mathrm{div}_p(g_t) + \mu_k \left(\tau_t + \frac{1}{2}\right)},$$
$$\tau^t = \frac{1}{\delta} \frac{\tau^{t-1} + \frac{1}{2}}{-\mathrm{div}_p(g_{t-1})} \cdot \lambda_{t-1}.$$

In the above, $p/|p|$ at $p = 0$ can be any fixed number and does not affect the performance of AMP.A. Further, the "divergence" term $\mathrm{div}_p(g_t)$ is defined as

$$\mathrm{div}_p(g_t) \triangleq \frac{1}{m} \sum_{a=1}^{m} \frac{1}{2} \left(\frac{\partial g(p_a^t, y_a)}{\partial p_a^R} - \mathrm{i} \frac{\partial g(p_a^t, y_a)}{\partial p_a^I} \right)$$
$$= \frac{1}{m} \sum_{a=1}^{m} \frac{y_a}{2|p_a^t|} - 1, \tag{I.3}$$

where p_a^R and p_a^I denote the real and imaginary parts of p_a^t respectively (i.e., $p_a^t = p_a^R + \mathrm{i} p_a^I$). The derivations of AMP.A can be found in Appendix A of the ArXiv preprint [26].

The first point that we would like to discuss here is the effect of the regularizer on AMP.A. For the moment suppose that the noise $\boldsymbol{w}$ is zero. Does including the regularizer in (I.2) benefit AMP.A? Clearly, any regularization may introduce unnecessary bias to the solution. Hence, if the final goal is to obtain $\boldsymbol{x}_*$ exactly we should set $\mu = 0$. However, the optimization problem in (I.2) is non-convex and iterative algorithms intended to solve it can get stuck at bad local minima. In this regard, regularization can still help AMP.A to escape bad local minima through continuation. Continuation is popular in convex optimization for improving the convergence rate of iterative algorithms [27]. In continuation we start with a value of μ for which AMP.A is capable of finding the global minimizer of (I.2). Then, once AMP.A converges we will either decrease or increase μ a little bit (depending on the final value of μ for which we want to solve the problem) and use the previous fixed point of AMP.A as the initialization for the new AMP.A. We continue this process until we reach the value of μ we are interested in. For instance, if we would like to solve the noiseless phase retrieval problem then μ should eventually go to zero so that we do not introduce unnecessary bias. The rationale behind continuation is the following. Let μ and μ' be two different values of the regularization parameter, and they are close to each other. Suppose that the global minimizer of (I.2) with regularization parameter μ' is $\boldsymbol{x}(\mu')$ and is given to the user. Suppose further that the user would like to find the global minimizer of (I.2) with μ. Then, it is conceivable that the global minimizer of the new problem is close to $\boldsymbol{x}(\mu')$.[1] Hence, the user can initialize AMP.A with $\boldsymbol{x}(\mu')$ and hope that the algorithm may converge to the global minimizer of (I.2) for μ.

A more general version of the continuation idea we discussed above is to let μ change at every iteration (denoted as μ^t), and set λ_t according to μ^t:

$$\lambda_t = \frac{-\mathrm{div}_p(g_t)}{-\mathrm{div}_p(g_t) + \mu^t \left(\tau_t + \frac{1}{2}\right)}, \tag{I.4}$$

This way we can not only automate the continuation process, but also let AMP.A decide which choice of μ is appropriate at a given stage of the algorithm. Our discussion so far has been heuristic. It is not clear whether and how much the generalized continuation can benefit the algorithm. To give a partial answer to this question we focus on the following particular continuation strategy: $\mu^t = \frac{1+2\mathrm{div}_p(g_t)}{1+2\tau_t}$ and obtain the following version of AMP.A:

$$\boldsymbol{p}^t = \boldsymbol{Ax}^t - \frac{2}{\delta} g(\boldsymbol{p}^{t-1}, \boldsymbol{y}), \tag{I.5a}$$
$$\boldsymbol{x}^{t+1} = 2\left[-\mathrm{div}_p(g_t) \cdot \boldsymbol{x}^t + \boldsymbol{A}^{\mathrm{H}} g(\boldsymbol{p}^t, \boldsymbol{y}) \right]. \tag{I.5b}$$

Below we informally discuss some of the results we will prove in this paper.

Informal Result 1: Consider the AMP.A algorithm for complex-valued models (where both $\boldsymbol{A}$ and $\boldsymbol{x}_\star$ are complex-valued) with $\mu^t = \frac{1+2\mathrm{div}_p(g_t)}{1+2\tau_t}$. Under the noiseless setting, if $\delta > \frac{64}{\pi^2} - 4 \approx 2.5$, then $\boldsymbol{x}^t$ "converges to" $\boldsymbol{x}_*$ as long as the initial estimate $\boldsymbol{x}^0$ is not orthogonal to $\boldsymbol{x}_*$ and $\|\boldsymbol{x}^0\| = \|\boldsymbol{x}_*\|$. When $2 < \delta < \frac{64}{\pi^2} - 4$, AMP.A has a fixed point at $\boldsymbol{x}_*$. However, it has to be initialized very carefully to reach $\boldsymbol{x}_*$.

Before we discuss and explain the implications of this result, let us expand the scope of our results. This extension enables us to compare our results with existing work [20]–[22]. So far,

[1]Given the sometimes complex geometry of non-convex problems, this might not always be the case.

we have discussed the case $\boldsymbol{x}_* \in \mathbb{C}^n$. However, in some applications, such as astronomical imaging, we are interested in real-valued signals $\boldsymbol{x}_* \in \mathbb{R}^n$. In Section III, we will introduce a real-valued version of AMP.A. The following informal result summarizes the performance of this algorithm.

Informal Result 2: Consider the AMP.A algorithm for real-valued signals with $\mu^t = \frac{2+2\text{div}_p(g_t)}{1+2\tau_t}$. Under the noiseless setting, if $\delta > \frac{\pi^2}{4} - 1 \approx 1.5$, then $\boldsymbol{x}^t$ "converges to" $\boldsymbol{x}_*$ as long as the initialization is not orthogonal to $\boldsymbol{x}_*$. When $1+\frac{4}{\pi^2} < \delta < \frac{\pi^2}{4}-1$, AMP.A has a fixed point at $\boldsymbol{x}_*$. However, it has to be initialized very carefully to reach $\boldsymbol{x}_*$.

We would like to make the following remarks about these two results:

1) As is clear from our second informal result, when $\delta < 1 + \frac{4}{\pi^2}$, AMP.A cannot converge to $\boldsymbol{x}_*$. This value of δ is different from the information theoretic lower bound $\delta = 1$ [28]. This discrepancy is in fact due to the type of continuation we used in this paper. Note that this issue does not happen in the complex-valued AMP.A. The search for a better continuation strategy for the real-valued AMP.A is left as future research.
2) Simulation results presented in our forthcoming paper [29] show that for real-valued signals, AMP.A with $\mu = 0$ can only recover when $\delta > 2.5$. As mentioned in our second informal result, continuation has improved the threshold of correct recovery to $\delta \approx 1.5$.
3) How much does spectral initialization improve the performance of AMP.A? To answer this question, let us focus on the real-valued signals. As discussed in our second Informal result, two values of δ are important for AMP.A: $\delta = \frac{\pi^2}{4} - 1 \approx 1.5$ and $\delta = 1 + \frac{4}{\pi^2} \approx 1.4$. If $\delta > 1.5$, then AMP.A recovers $\boldsymbol{x}_*$ exactly as long as the initialization is not orthogonal to $\boldsymbol{x}_*$. In this case spectral method helps, since it offers an initialization that is not orthogonal to $\boldsymbol{x}_*$. However, if the mean of $\boldsymbol{x}_*$ is not zero, a simple initial estimate $\mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}}$ can work as well as the spectral initialization. Hence, in this case spectral initialization does not offer a major improvement. A more important question is whether spectral initialization can help AMP.A to perform exact recovery for $\delta < 1.5$. Our forthcoming paper [29] shows that the answer to this question is negative. Hence, as long as the final estimate of AMP.A is concerned, the impact of spectral initialization seems to be marginal.

Now let us discuss the performance of AMP.A under noisy settings. We assume that the measurement noise is Gaussian and small. Clearly, in this setting exact recovery is impossible, hence we study the asymptotic mean square error defined as the following almost sure limit ($\theta_t \triangleq \angle \frac{1}{n}\langle \boldsymbol{x}_*, \boldsymbol{x}^t \rangle$)

$$\text{AMSE}(\delta, \sigma_w^2) \triangleq \lim_{t\to\infty} \lim_{n\to\infty} \frac{\|\boldsymbol{x}^t - e^{\mathrm{i}\theta_t}\boldsymbol{x}_*\|_2^2}{n}, \tag{I.6}$$

Informal Result 3: Consider the AMP.A algorithm for complex-valued signals with $\mu^t = \frac{1+2\text{div}_p(g_t)}{1+2\tau_t}$. Let $\delta > \frac{64}{\pi^2} - 4 \approx 2.5$, then

$$\lim_{\sigma_w^2 \to 0} \frac{\text{AMSE}(\delta, \sigma_w^2)}{\sigma_w^2} = \frac{4}{1 - \frac{2}{\delta}}. \tag{I.7}$$

Notice that the above result was derived based under the assumption $\mathbb{E}[|A_{ai}|^2] = 1/m$. To interpret the above result correctly, we should discuss the signal to noise ratio of each measurement. Suppose that $\frac{1}{n}\|\boldsymbol{x}_*\|^2 = 1$. Then the signal to noise ratio of each measurement is $\mathbb{E}\left[\left|\sum_i A_{ai}x_{*,i}\right|^2\right]/\sigma_w^2 = \frac{1}{\delta\sigma_w^2}$. In other words, as we increase the number of measurements or equivalently δ, then we reduce the signal to noise ratio of each measurement too. This causes some issues when we compare the $\text{AMSE}(\delta, \sigma_w^2)$ for different values of δ. One easy fix is to assume that the variance of the noise is $\sigma_w^2 = \frac{\tilde{\sigma}_w^2}{\delta}$, where $\tilde{\sigma}_w^2$ is a fixed number. Then we can define the noise sensitivity as

$$\text{NS}(\tilde{\sigma}_w^2, \delta) = \frac{\text{AMSE}(\delta, \sigma_w^2)}{\tilde{\sigma}_w^2}.$$

It is straightforward to use (I.7) to show that $\text{NS}(\tilde{\sigma}_w, \delta) = \frac{4}{\delta-2}$. Note that if we use AMP.A with $\delta \approx \delta_{\text{AMP}}$, then the noise sensitivity is approximately 8. If this level of noise sensitivity is not acceptable for an application, then the user should collect more measurements to reduce the noise sensitivity. Noise sensitivity can also be calculated for real-valued AMP.A:

Informal Result 4: Consider the AMP.A algorithm for real-valued signals with $\mu^t = \frac{2+2\text{div}_p(g_t)}{1+2\tau_t}$. Let $\delta > \frac{\pi^2}{4} - 1 \approx 1.5$, then

$$\lim_{\sigma_w^2 \to 0} \frac{\text{AMSE}(\delta, \sigma_w^2)}{\sigma_w^2} = \frac{1}{\left(1 + \frac{4}{\pi^2}\right)^{-1} - \frac{1}{\delta}}.$$

B. Related Work

1) Existing Theoretical Work: Early theoretical results on phase retrieval, such as PhaseLift [1] and PhaseCut [30], are based on semidefinite relaxations. For random Gaussian measurements, a variant of PhaseLift can recover the signal exactly (up to global phase) in the noiseless setting using $O(n)$ measurements [31]. However, PhaseLift (or PhaseCut) involves solving a semidefinite programming (SDP) and is computationally prohibitive for large-scale applications. A different convex optimization approach for phase retrieval, which has the same $O(n)$ sample complexity, was independently proposed in [8] and [9]. This method is formulated in the natural signal space and does not involve lifting, and is therefore computationally more attractive than SDP-based counterparts. However, both methods require an anchor vector that has non-zero correlation with the true signal, and the quality of the recovery highly depends on the quality of the anchor.

Apart from convex relaxation approaches, non-convex optimization approaches attract considerable recent interests. These algorithms typically consist of a carefully designed initialization step (usually accomplished via a spectral method [2]) followed by iterations that refine the estimate.

An early work in this direction is the alternating minimization algorithm proposed in [2], which has sub-optimal sample complexity. Another line of work includes the Wirtinger flow algorithm [4], [32], truncated Wirtinger flow algorithm [5], and other variants [6], [7], [10], [12]. Other approaches include Kaczmarz method [16], [17], [33], [34], trust region method [11], coordinate decent [18], prox-linear algorithm [13], Polyak subgradient method [15], block coordinate descent [35] and PhaseLin algorithm [36].

Most of the above theoretical results guarantee successful recovery with $m = \delta n$ measurements (or more) where δ is a fixed often large constant. However, such theories are not capable of providing fair comparison among different algorithms. To resolve this issue researchers have started studying the performance of different algorithms under the asymptotic setting $m/n \to \delta$ and $n \to \infty$. In [8], it is proved that the PhaseMax algorithm succeeds when $\delta > \frac{4}{\alpha}$ (for complex-valued models), where α is a constant that depends on the angle between of anchor vector and the true signal vector. An interesting iterative projection method was proposed in [37], whose dynamics can be characterized exactly under this asymptotic setting. However, [37] does not analyze the number of measurements required for this algorithm to work. The work in [14] provides sharp characterization of the spectral initialization step (which is a key ingredient to many of the above algorithms). The analysis in [14] reveals a phase transition phenomenon: spectral method produces an estimate not orthogonal to the signal if and only if δ is larger than a threshold (called "weak threshold" in [19]). Later, Mondelli and Montanari [19] derived the information-theoretically optimal weak threshold (which is 0.5 for the real-valued model and 1 for the complex-valued model) and proved that the optimal weak threshold can be achieved by an optimally-tuned spectral method. Using the non-rigorous replica method from statistical physics, Dhifallah and Lu [20] analyze the exact threshold of δ (for the real-value setting) above which the PhaseMax method in [8] and [9] achieves perfect recovery. The analysis in [20] shows that the performance of PhaseMax highly depends on initialization (see [20, Fig. 1]), and the required δ is lower bounded by 2 for real-valued models. On the other hand, AMP.A proposed in this paper achieves perfect recovery for $\delta > 1.5$ under the same setting. The analysis in [20] was later rigorously proved in [21] via the Gaussian min-max framework [38], [39], and a new algorithm called PhaseLamp was proposed. The PhaseLamp method has superior recovery performance over PhaseMax, but again it does not work when $\delta < 2$ for real-valued models. Further, Dhifallah and Lu [20] and Dhifallah *et al.* [21] focus on the noiseless scenario, while in this paper we also analyze the noise sensitivity of AMP.A. Finally, a recent paper [22] derived an upper bound of δ such that PhaseLift achieves perfect recovery. The exact value of this upper bound can be derived by solving a three-variable convex optimization problem and empirically [22] shows that $\delta \approx 3$ for real-valued models.

2) Existing Work Based on AMP: Our work in this paper is based on the approximate message passing (AMP) framework [40], [41], in particular the generalized approximate message passing (GAMP) algorithm developed and analyzed in [25] and [42]. A key property of AMP (including GAMP) is that its asymptotic behavior can be characterized exactly via the state evolution platform [25], [40]–[42].

For phase retrieval, a Bayesian GAMP algorithm has been proposed in [43]. However, [43] did not provide rigorous performance analysis, partly due to the heuristic treatments used in the algorithm (such as damping and restart). Another work related to ours is the recent paper [28] (appeared on Arxiv while we were preparing this paper), which analyzed the phase transitions of the Bayesian GAMP algorithms for a class of nonlinear acquisition models. For the phase retrieval problem, a phase transition diagram was shown in [28, Fig. 1] under a Bernoulli-Gaussian signal prior. The numerical results in [28] indeed achieve state-of-the-art reconstruction results for real-valued models. However, [28] did not provide the analysis of their results and in particular did not mention how they handle a difficulty related to initialization. Further, the algorithm in [28] is based on the Bayesian framework which assumes that the signal and the measurements are generated according to some known distributions. Contrary to [28] and [43], this paper considers a version of GAMP derived from solving the popular optimization problem (I.2). We provide rigorous performance analysis of our algorithm for both real and complex-valued models. Note that the advantages and disadvantages of Bayesian and optimization-based techniques have been a long debate in the field of Statistics. Hence, we do not repeat those debates here. Given our experience in the fields of compressed sensing and phase retrieval, it seems that the performance of Bayesian algorithms are more sensitive to their assumptions than the optimization-based schemes. Furthermore, performance analyses of Bayesian algorithms are often very challenging under "non-ideal" situations which the algorithms are not designed for.

Here, we emphasize another advantage of our approach. Given the fact that the most popular schemes in practice are iterative algorithms derived for solving non-convex optimization problems, the detailed analyses of AMP.A presented in our paper may also shed light on the performance of these algorithms and suggest new ideas to improve their performances.

3) Fundamental Limits: It the literature of phase retrieval, it is well known that to make the signal-to-observation mapping injective one needs at least $m = 4n$ measurements [44] (or $m = 2n$ [45] in the case of real-valued models). On the other hand, the measurement thresholds obtained in this paper are $\delta = \frac{64}{\pi^2} - 4 \approx 2.5$ and $\delta = \frac{\pi^2}{4} - 1 \approx 1.5$ respectively. In fact, our algorithm can in principle recover the signal when $\delta > 2$ and $\delta > 1 + \frac{4}{\pi^2}$ (or $\delta > 1$ if continuation is not applied) for complex and real-valued models, provided that the algorithm is initialized close enough to the signal (though no known initialization strategy can accomplish this goal). Hence, our threshold are even smaller than the injectivity bounds. We emphasize that this is possible since the injectivity bounds derived in [44] and [45] are defined for *all* $\boldsymbol{x}_*$ (which can depend on $\boldsymbol{A}$ in the worst case scenario). This is different from our assumption that $\boldsymbol{x}_*$ is independent of $\boldsymbol{A}$, which is more relevant in applications where one has some freedom to randomize the sampling mechanism. In fact, several papers have observed that their algorithm can operate at the injectivity thresholds $\delta = 2$ for real-valued models [6], [13]. These two different notions of thresholds were discussed in [46]. In the context of phase retrieval, the reader is referred to the recent paper [47], which showed that by solving a compression-based optimization problem, the required number of observations for recovery is essentially the information dimension of the signal (see [47] for the precise definition). For instance, if the signal is k-sparse and complex-valued, then $2k$ measurements suffice.

C. Organization of the Paper

The structure of the rest of the paper is as follows: Section II mentions the asymptotic framework of the paper, and summarizes our main results on the asymptotic analysis of AMP.A. Section III discusses the real-valued AMP.A algorithm and its analysis. Section IV explains what have been discussed in the conference version of this paper [48], and points out several possible future directions. Sections V and VI present the proofs of our main results. Finally, the appendix summarizes some properties of elliptic integrals that are used throughout this paper.

D. Notations

$\bar{a}$ denotes the conjugate of a complex number a. $\angle a$ denotes the phase of a. We use bold lower-case and upper case letters for vectors and matrices respectively. For a matrix $\boldsymbol{A}$, $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{H}}$ denote the transpose of a matrix and its Hermitian respectively. Throughout the paper, we also use the following two notations: $\mathbf{1} \triangleq [1, \ldots, 1]^{\mathrm{T}}$ and $\mathbf{0} \triangleq [0, \ldots, 0]^{\mathrm{T}}$. $\phi(x)$ and $\Phi(x)$ are used for the probability density function and cumulative distribution function of the standard Gaussian random variable. A random variable a said to be circularly-symmetric Gaussian, denoted as $a \sim \mathcal{CN}(0, \sigma^2)$, if $a = a_R + \mathrm{i} a_I$ and a_R and a_I are two independent real Gaussian random variables with mean zero and variance $\sigma^2/2$. Finally, we define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \triangleq \sum_{i=1} \bar{a}_i b_i$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^d$.

II. Asymptotic Analysis of AMP.A

In this section, we present the asymptotic platform under which AMP.A is studied, and we derive a set of equations, known as state evolution (SE), that capture the performance of AMP.A under the asymptotic analysis.

A. Asymptotic Framework and State Evolution

Our analysis of AMP.A is carried out based on a standard asymptotic framework developed in [41] and [49]. In this framework, we let $m, n \to \infty$, while $m/n \to \delta$. Within this section, we will write $\boldsymbol{x}_*$, $\boldsymbol{x}^t$, $\boldsymbol{w}$ and $\boldsymbol{A}$ as $\boldsymbol{x}_*(n)$, $\boldsymbol{x}^t(n)$, $\boldsymbol{w}(n)$ and $\boldsymbol{A}(n)$ to make explicit their dependency on the signal dimension n. In this section we focus on the complex-valued AMP. We postpone the discussion of the real-valued AMP until Section III. Following [50], we introduce the following definition of converging sequences.

Definition 1: The sequence of instances $\{\boldsymbol{x}_*(n), \boldsymbol{A}(n), \boldsymbol{w}(n)\}$ is said to be a converging sequence if the following hold:

- $\frac{m}{n} \to \delta \in (0, \infty)$, *as* $n \to \infty$.
- $\boldsymbol{A}(n)$ *has i.i.d. Gaussian entries where* $A_{ij} \sim \mathcal{CN}(0, 1/m)$.
- *The empirical distribution of* $\boldsymbol{x}_*(n) \in \mathbb{C}^n$ *converges weakly to a probability measure* p_X *with bounded second moment. Further,* $\frac{1}{n}\|\boldsymbol{x}_*(n)\|^2 \to \kappa^2$ *where* $\kappa^2 \in (0, \infty)$ *is the second moment of* p_X*. For convenience and without loss of generality, we assume* $\kappa = 1$.[2]
- *The empirical distribution of* $\boldsymbol{w}(n) \in \mathbb{C}^n$ *converges weakly to* $\mathcal{CN}(0, \sigma_w^2)$.

Under the asymptotic framework introduced above, the behavior of AMP.A can be characterized exactly. Roughly speaking, the estimate produced by AMP.A in each iteration is approximately distributed as the (scaled) true signal + additive Gaussian noise; in other words, $\boldsymbol{x}^t$ can be modeled as $\alpha_t \boldsymbol{x}_* + \sigma_t \boldsymbol{h}$, where $\boldsymbol{h}$ behaves like an iid standard complex normal noise. We will clarify this claim in Theorem 1 below. The scaling constant α_t and the noise standard deviation σ_t evolve according to a known deterministic rule, called the state evolution (SE), defined below.

Definition 2: Starting from fixed $(\alpha_0, \sigma_0^2) \in \mathbb{C} \times \mathbb{R}_+ \backslash (0, 0)$, the sequences $\{\alpha_t\}_{t\geq 1}$ and $\{\sigma_t^2\}_{t\geq 1}$ are generated via the following recursion:

$$\begin{aligned} \alpha_{t+1} &= \psi_1(\alpha_t, \sigma_t^2), \\ \sigma_{t+1}^2 &= \psi_2(\alpha_t, \sigma_t^2; \delta, \sigma_w^2), \end{aligned} \tag{II.1}$$

where $\psi_1 : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}$ and $\psi_2 : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ are respectively given by

$$\begin{aligned} \psi_1(\alpha, \sigma^2) &= 2 \cdot \mathbb{E}\left[\partial_z g(P, Y)\right] = \mathbb{E}\left[\frac{\bar{Z} P}{|Z|\,|P|}\right], \\ \psi_2(\alpha, \sigma^2; \delta, \sigma_w^2) &= 4 \cdot \mathbb{E}\left[|g(P, Y)|^2\right] \\ &= 4 \cdot \mathbb{E}\left[(|P| - |Z| - W)^2\right]. \end{aligned}$$

In the above equations, the expectations are over all random variables involved: $Z \sim \mathcal{CN}(0, 1/\delta)$, $P = \alpha Z + \sigma B$ where $B \sim \mathcal{CN}(0, 1/\delta)$ is independent of Z, and $Y = |Z| + W$ where $W \sim \mathcal{CN}(0, \sigma_w^2)$ is independent of both Z and B. Further, the partial Wirtinger derivative $\partial_z g(p, |z| + w)$ is defined as:

$$\partial_z g(p, |z|+w) \triangleq \frac{1}{2}\left[\frac{\partial}{\partial z_R} g(p, |z| + w) - \mathrm{i}\frac{\partial}{\partial z_I} g(p, |z|+w)\right],$$

where z_R and z_I are the real and imaginary parts of z (i.e., $z = z_R + \mathrm{i} z_I$).

Remark 1: The functions ψ_1 and ψ_2 are well defined except when both α and σ^2 are zero.

Remark 2: Most of the analysis in this paper is concerned with the noiseless case. For brevity, we will often write $\psi_2(\alpha, \sigma; \delta, 0)$ (where $\sigma_w^2 = 0$) as $\psi_2(\alpha, \sigma; \delta)$. Further, when

[2] Otherwise, we can introduce the following normalized variables: $\tilde{\boldsymbol{y}} = \boldsymbol{y}/\kappa$, $\tilde{\boldsymbol{x}} = \boldsymbol{x}/\kappa$, $\tilde{\boldsymbol{w}} = \boldsymbol{w}/\kappa$, $\tilde{\boldsymbol{x}}^t = \boldsymbol{x}^t/\kappa$ and $\tilde{\boldsymbol{p}}^t = \boldsymbol{p}^t/\kappa$. One can verify that the AMP.A algorithm defined in (I.5) for these normalized variables remains unchanged. Therefore, we can view that our analyses are carried out for these normalized variables; we don't need to actually change the algorithm though.

our focus is on α and σ^2 rather than δ, we will simply write $\psi_2(\alpha, \sigma^2; \delta)$ as $\psi_2(\alpha, \sigma^2)$.

In [26, Appendix B], we simplify the functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ into the following expressions (with θ_α being the phase of α):

$$\psi_1(\alpha, \sigma^2) = e^{i\theta_\alpha} \cdot \int_0^{\frac{\pi}{2}} \frac{|\alpha| \sin^2\theta}{\left(|\alpha|^2 \sin^2\theta + \sigma^2\right)^{\frac{1}{2}}} d\theta, \quad \text{(II.2a)}$$

$$\psi_2(\alpha, \sigma^2; \delta, \sigma_w^2) \quad \text{(II.2b)}$$

$$= \frac{4}{\delta}\left(|\alpha|^2 + \sigma^2 + 1 - \int_0^{\frac{\pi}{2}} \frac{2|\alpha|^2 \sin^2\theta + \sigma^2}{\left(|\alpha|^2 \sin^2\theta + \sigma^2\right)^{\frac{1}{2}}} d\theta\right) + 4\sigma_w^2. \quad \text{(II.2c)}$$

The above expressions for ψ_1 and ψ_2 are more convenient for our analysis.

The state evolution framework for generalized AMP (GAMP) algorithms [25] was first introduced and analyzed in [25] and later formally proved in [42]. As we will show later in Theorem 1, SE characterizes the macroscopic behavior of AMP.A. To apply the results in [25] and [42] to AMP.A, however, we need two generalizations. First, we need to extend the results in [25] and [42] to complex-valued models. This is straightforward by applying a complex-valued version of the conditioning lemma introduced in [25] and [42]. Second, existing results in [25] and [42] require the function g to be smooth. Our simulation results in case of complex-valued AMP.A show that SE predicts the performance of AMP.A despite the fact that g is not smooth. Since our paper is long, we postpone the proof of this claim to another paper. Instead we use the smoothing idea discussed in [24] to connect the SE equations presented in (II.1) with the iterations of AMP.A in (I.5). Let $\epsilon > 0$ be a small fixed number. Consider the following smoothed version of AMP.A:

$$\boldsymbol{p}^t = \boldsymbol{A}\boldsymbol{x_\epsilon}^t - \frac{2}{\delta} g_\epsilon(\boldsymbol{p}^{t-1}, \boldsymbol{y}),$$
$$\boldsymbol{x_\epsilon}^{t+1} = 2\left[-\text{div}_p(g_{t,\epsilon}) \cdot \boldsymbol{x_\epsilon}^t + \boldsymbol{A}^{\mathrm{H}} g_\epsilon(\boldsymbol{p}^t, \boldsymbol{y})\right],$$

where $g_\epsilon(\boldsymbol{p}^{t-1}, \boldsymbol{y})$ refers to a vector produced by applying $g_\epsilon : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}$ below component-wise:

$$g_\epsilon(p, y) \triangleq y \cdot h_\epsilon(p) - p,$$

where for $p = p_1 + i p_2$, $h_\epsilon(p)$ is defined as

$$h_\epsilon(p) \triangleq \frac{p_1 + i p_2}{\sqrt{p_1^2 + p_2^2 + \epsilon}}.$$

Note that as $\epsilon \to 0$, $g_{t,\epsilon} \to g_t$ and hence we expect the iterations of smoothed-AMP.A converge to the iterations of AMP.A.

Theorem 1 (Asymptotic Characterization): Let $\{\boldsymbol{x}_*(n), \boldsymbol{A}(n), \boldsymbol{w}(n)\}$ be a converging sequence of instances. For each instance, let $\boldsymbol{x}^0(n)$ be an initial estimate independent of $\boldsymbol{A}(n)$. Assume that the following hold almost surely

$$\lim_{n\to\infty} \frac{1}{n}\langle \boldsymbol{x}_*, \boldsymbol{x}^0 \rangle = \alpha_0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{n}\|\boldsymbol{x}^0\|^2 = \sigma_0^2 + |\alpha_0|^2.$$

Let $\boldsymbol{x}_\epsilon^t(n)$ be the estimate produced by the smoothed AMP.A *initialized by $\boldsymbol{x}^0(n)$ (which is independent of $\boldsymbol{A}(n)$) and $\boldsymbol{p}^{-1}(n) = \boldsymbol{0}$. Let $\epsilon_1, \epsilon_2, \ldots$ denote a sequence of smoothing parameters for which $\epsilon_i \to 0$ as $i \to \infty$ Then, for any iteration $t \geq 1$, the following holds almost surely*

$$\lim_{j\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n |x_{\epsilon_j,i}^t(n) - e^{i\theta_t} x_{*,i}|^2$$
$$= \mathbb{E}\left[|X^t - e^{i\theta_t} X_*|^2\right] = \left|1 - |\alpha_t|\right|^2 + \sigma_t^2, \quad \text{(II.3)}$$

where $\theta_t = \angle\alpha_t$, $X^t = \alpha_t X_* + \sigma_t H$ and $X_* \sim p_X$ is independent of $H \sim \mathcal{CN}(0, 1)$. Further, $\{\alpha\}_{t\geq 1}$ and $\{\sigma_t^2\}_{t\geq 1}$ are determined by (II.1) *with initialization α_0 and σ_0^2.*

Proof: Since the proof of the real-valued and complex valued signals look similar, for the sake of notational simplicity we present the proof for the real-valued signals. First note that according to [19, Lemma 13][3] for the smoothed AMP.A algorithm we know that almost surely

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left(x_{\epsilon_j,i}^{t+1}(n) - \text{sign}(\alpha_t) \cdot x_{*,i}\right)^2$$
$$= \mathbb{E}(X_{\epsilon_j}^{t+1} - \text{sign}(\alpha_t) \cdot X_*)^2,$$

where $X_\epsilon^t = \alpha_{\epsilon,t} X_* + \sigma_{\epsilon,t} H$ and $X_* \sim p_X$ is independent of $H \sim \mathcal{N}(0, 1)$, and $\alpha_{\epsilon,t}$ and $\sigma_{\epsilon,t}$ satisfy the following iterations:

$$\alpha_{\epsilon,t+1} = \mathbb{E}\left[\partial_z g_\epsilon(P^t, Y)\right],$$
$$\sigma_{\epsilon,t+1}^2 = \mathbb{E}[g_\epsilon^2(P^t, Y)],$$

where $Y = |Z| + W$, $P^t = \alpha_{\epsilon,t} Z + \sigma_{\epsilon,t} B$, where $B \sim \mathcal{N}(0, 1/\delta)$ is independent of $Z \sim \mathcal{N}(0, 1/\delta)$ and $W \sim \mathcal{N}(0, 1/\delta)$. It is also straightforward to use an induction step similar to the one presented in the proof of [24, Th. 1] and show that $(\alpha_{\epsilon,t}, \sigma_{\epsilon,t}^2) \to (\alpha_t, \sigma_t^2)$ as $i \to \infty$, where (α_t, σ_t^2) satisfy

$$\alpha_{t+1} = \mathbb{E}\left[\partial_z g(P^t, Y)\right],$$
$$\sigma_{t+1}^2 = \mathbb{E}[g^2(P^t, Y)].$$

□

B. Convergence of the SE for Noiseless Model

We now analyze the dynamical behavior of the SE. Before we proceed, we point out that in phase retrieval, one can only hope to recover the signal up to global phase ambiguity [1], [2], [4], for generic signals without any structure. In light of (II.3), AMP.A is successful if $|\alpha_t| \to 1$ and $\sigma_t^2 \to 0$ as $t \to \infty$.

Let us start with the following interesting feature of the state evolution, which can be seen from (II.2).

Lemma 1: For any $(\alpha_0, \sigma_0^2) \in \mathbb{C} \times \mathbb{R}_+ \backslash (0, 0)$, ψ_1 and ψ_2 satisfy the following properties:

(i) *$\psi_1(\alpha, \sigma^2) = \psi_1(|\alpha|, \sigma^2) \cdot e^{i\theta_\alpha}$, with $e^{i\theta_\alpha}$ being the phase of α;*

(ii) *$\psi_2(\alpha, \sigma^2) = \psi_2(|\alpha|, \sigma^2)$.*

Hence, if θ_t denotes the phase of α_t, then $\theta_t = \theta_0$.

[3] The proof for a more general result was first presented in [42]. However, we found [19] easier to follow. The reader may also find [25, Claim 1] and related discussions useful, although no formal proof was provided.

In light of this lemma, we can focus on real and nonnegative values of α_t. In particular, we assume that $\alpha_0 \geq 0$ and we are interested in whether and under what conditions can the SE converge to the fixed point $(\alpha, \sigma^2) = (1, 0)$. The following two values of δ will play critical roles in the analysis of SE:

$$\delta_{\text{AMP}} \triangleq \frac{64}{\pi^2} - 4 \approx 2.5,$$
$$\delta_{\text{global}} \triangleq 2.$$

Our next theorem reveals the importance of δ_{AMP}. The proof of this theorem detailed in Section V.

Theorem 2 (Convergence of SE): Consider the noiseless model where $\sigma_w^2 = 0$. If $\delta > \delta_{\text{AMP}}$, then for any $0 < \alpha_0 \leq 1$ and $\sigma_0^2 \leq 1$, the sequences $\{\alpha_t\}_{t \geq 1}$ and $\{\sigma_t^2\}_{t \geq 1}$ defined in (II.1) *converge to*

$$\lim_{t\to\infty} \alpha_t = 1 \quad \text{and} \quad \lim_{t\to\infty} \sigma_t^2 = 0.$$

There are a couple of points that we would like to emphasize here:

1) $0 < \alpha_0 \leq 1$ and $\sigma_0^2 \leq 1$ is a pessimistic condition for Theorem 2. In particular, when $\delta > 4$, this condition could be relaxed to $\alpha_0 \neq 0$ and $\sigma_0^2 < \infty$. In this paper, we did not try to optimize this condition since it is fairly loose and can be achieved by the spectral method in the noiseless case. In other words, if $\delta > \delta_{\text{AMP}}$ the issue of initialization becomes minor. We will report more details of this claim in [29]. Alternatively, $0 < \alpha_0 \leq 1$ and $\sigma_0^2 \leq 1$ can also be achieved for the noiseless setting if the signal of interest has nonzero mean. To see this, consider the initialization $\boldsymbol{x}^0 = \mathbf{1}$. (In the general case where $\kappa \neq 1$, we initialize as $\boldsymbol{x}^0 = \kappa\mathbf{1}$. Note that $\kappa^2 = \|\boldsymbol{x}\|^2/n$ can be accurately estimated in the noiseless setting [14].) Such initialization ensures that $\alpha_0^2 + \sigma_0^2 = 1$. Further, $\alpha_0 = \mathbb{E}[X_*] \neq 0$. Therefore, $\alpha_0 \in (0, 1)$ and $\sigma_0^2 \in (0, 1)$.
2) $\alpha_0 \neq 0$ is essential for the success of AMP.A. This can be seen from the fact that $\alpha = 0$ is always a fixed point of $\psi_1(\alpha, \sigma^2)$ for any $\sigma^2 > 0$. From our definition of α_0 in Theorem 1, $\alpha_0 = 0$ is equivalent to $\frac{1}{n}\langle \boldsymbol{x}_*, \boldsymbol{x}^0 \rangle = 0$. This means that the initial estimate $\boldsymbol{x}^0$ cannot be orthogonal to the true signal vector $\boldsymbol{x}_*$, otherwise there is no hope to recover the signal no matter how large δ is.
3) In the current paper, we did not analyze the convergence rate of the state evolution. Empirically, we observed that typically the SE converges exponentially fast after a few iterations. Rigorously proving this finding is left as possible future work.

The following theorem describes the importance of δ_{global} and its proof can be found in Section VI.

Theorem 3 (Local Convergence of SE): When $\sigma_w^2 = 0$, then $(\alpha, \sigma^2) = (1, 0)$ is a fixed point of the SE in (II.2). *Furthermore, if $\delta > \delta_{\text{global}}$, then there exist two constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the SE converges to this fixed point for any $\alpha_0 \in (1 - \epsilon_1, 1)$ and $\sigma_0^2 \in (0, \epsilon_2)$. On the other hand if $\delta < \delta_{\text{global}}$, then the SE cannot converge to* $(1, 0)$ *except when initialized there.*

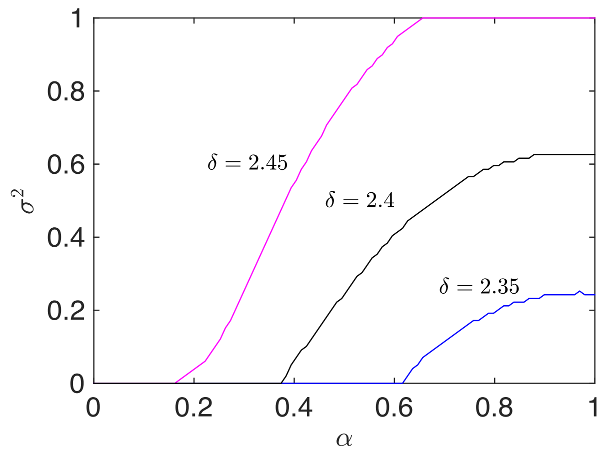

Fig. 1. The regions below the curves exhibit the basin of attraction of $(\alpha, \sigma^2) = (1, 0)$. From left to right $\delta = 2.45$, $\delta = 2.4$, $\delta = 2.35$. The results are obtained by running the state evolution (SE) of AMP.A (complex-valued version) with α_0 and σ_0^2 chosen from 100×100 values equispaced in $[0, 1] \times [0, 1]$. The iteration number is $T = 5000$ and the reconstruction is considered successful is $|\alpha_T - 1| < 10^{-5}$ and $\sigma_T^2 < 10^{-10}$.

According to Theorem 3, with proper initialization, SE can potentially converge to (α, σ^2) even if $\delta_{\text{global}} < \delta < \delta_{\text{AMP}}$. However, there are two points we should emphasize here: (i) we find that when $\delta < \delta_{\text{AMP}}$, standard initialization techniques, such as the spectral method, do not help AMP.A converge to $\boldsymbol{x}_*$. We refer the reader to the conference version of this paper [48] for details. Hence, the question of finding initialization in the basin of attraction of $(\alpha, \sigma^2) = (1, 0)$ (when $\delta < \delta_{\text{AMP}}$) remains open for future research. (ii) As δ decreases from δ_{AMP} to δ_{global} the basin of attraction of $(\alpha, \sigma^2) = (1, 0)$ shrinks. Check the numerical results in Figure 1.

C. Noise Sensitivity

So far we have only discussed the performance of AMP.A in the ideal setting where the noise is not present in the measurements. In general, one can use (II.1) to calculate the asymptotic MSE (AMSE) of AMP.A as a function of the variance of the noise and δ. However, as our next theorem demonstrates it is possible to obtain an explicit and informative expression for AMSE of AMP.A in the high signal-to-noise ratio (SNR) regime.

Theorem 4 (Noise Sensitivity): Suppose that $\delta > \delta_{\text{AMP}} = \frac{64}{\pi^2} - 4$ and $0 < |\alpha_0| \leq 1$ and $\sigma_0^2 < 1$. Then, in the high SNR regime the asymptotic MSE defined in (I.6) *behaves as*

$$\lim_{\sigma_w^2 \to 0} \frac{\text{AMSE}(\sigma_w^2, \delta)}{\sigma_w^2} = \frac{4}{1 - \frac{2}{\delta}}.$$

The proof of this theorem can be found in [26, Appendix E].

III. Extension to Real-Valued Signals

Until now our focus is on complex-valued signals. In this section, our goal is to extend our results to real-valued signals. Since most of the results are similar to the complex-valued case, we will skip the details and only emphasize on the main differences.

A. AMP.A *Algorithm*

In the real-valued case, AMP.A uses the following iterations:

$$\begin{aligned} \boldsymbol{x}^{t+1} &= -\mathrm{div}_p(g_t)\cdot \boldsymbol{x}^t + \boldsymbol{A}^{\mathrm{T}} g(\boldsymbol{p}^t, \boldsymbol{y}), \\ \boldsymbol{p}^t &= \boldsymbol{A}\boldsymbol{x}^t - \frac{1}{\delta} g(\boldsymbol{p}^{t-1}, \boldsymbol{y}), \end{aligned} \tag{III.1a}$$

where $g(p, y) : \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$ is given by

$$g(p, y) \overset{\Delta}{=} y \cdot \mathrm{sign}(p) - p, \tag{III.1b}$$

where $\mathrm{sign}(p)$ denotes the sign of p. We emphasize that the divergence term $\mathrm{div}_p(g_t)$ contains a Dirac delta at 0 due to the discontinuity of the sign function. This makes the calculation of the divergence in the AMP.A algorithm tricky. One can use the smoothing idea we discussed in Section II-A. Alternatively, there are several possible approaches to estimate the divergence term. These practical issues will be discussed in details in our follow-up paper [29].

B. Asymptotic Analysis

Our analysis is based on the same asymptotic framework detailed in Section II-B. The only difference is that the measurement matrix is now real Gaussian with $A_{ij} \sim \mathcal{N}(0, 1/m)$ and $w_a \sim \mathcal{N}(0, \sigma_w^2)$. In the real-valued setting, the state evolution (SE) recursion of AMP.A in (III.1) becomes the following.

Definition 3: Starting from fixed $(\alpha_0, \sigma_0^2) \in \mathbb{R}\times\mathbb{R}_+\backslash(0, 0)$ *the sequences* $\{\alpha_t\}_{t\geq 1}$ *and* $\{\sigma_t^2\}_{t\geq1}$ *are generated via the following iterations:*

$$\begin{aligned} \alpha_{t+1} &= \psi_1(\alpha_t, \sigma_t^2), \\ \sigma_{t+1}^2 &= \psi_2(\alpha_t, \sigma_t^2; \delta, \sigma_w^2), \end{aligned} \tag{III.2}$$

where, with some abuse of notations, $\psi_1 : \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$ *and* $\psi_2 : \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}_+$ *are now defined as*

$$\psi_1(\alpha, \sigma^2) = \mathbb{E}[\partial_z g(P, |Y|)] = \mathbb{E}[\mathrm{sign}(ZP)],$$
$$\psi_2(\alpha, \sigma^2; \delta, \sigma_w^2) = \mathbb{E}[g^2(P, |Y|)] = \mathbb{E}\left[(|Z| - |P| + W)^2\right].$$

The expectations are over the following random variables: $Z \sim \mathcal{N}(0, 1/\delta)$*,* $P = \alpha Z + \sigma B$ *where* $B \sim \mathcal{N}(0, 1/\delta)$ *is independent of* Z*, and* $Y = |Z| + W$ *where* $W \sim \mathcal{N}(0, \sigma_w^2)$ *independent of both* Z *and* B*.*

In [26, Appendix D], we derived the following closed-form expressions of ψ_1 and ψ_2:

$$\psi_1(\alpha, \sigma^2) = \frac{2}{\pi}\arctan\left(\frac{\alpha}{\sigma}\right), \tag{III.3a}$$

$$\psi_2(\alpha, \sigma^2; \delta, \sigma_w^2) = \frac{1}{\delta}\left[\alpha^2 + \sigma^2 + 1 - \frac{4\sigma}{\pi} - \frac{4\alpha}{\pi}\arctan\left(\frac{\alpha}{\sigma}\right)\right] + \sigma_w^2. \tag{III.3b}$$

As in the complex-valued case, we would like to study the dynamics of these two equations. The following lemma simplifies the analysis.

Lemma 2: $\psi_1\left(\alpha, \sigma^2\right)$ *and* $\psi_2(\alpha, \sigma^2)$ *in* (III.3) *and* (III.3b) *have the following properties:*

(i) $\psi_1(\alpha, \sigma^2) = \psi_1(|\alpha|, \sigma^2)\cdot \mathrm{sign}(\alpha)$.

(ii) $\psi_2(\alpha, \sigma^2) = \psi_2(|\alpha|, \sigma^2)$.

Again the following two values of δ play a critical role in the performance of AMP:

$$\delta_{\mathrm{AMP}} = \frac{\pi^2}{4} - 1 \approx 1.47,$$
$$\delta_{\mathrm{global}} = 1 + \frac{4}{\pi^2} \approx 1.40.$$

The following two theorems correspond to Theorems 2 and 3 that explain the dynamics of SE for complex-valued signals. The proofs can be found in [26, Appendix D].

Theorem 5 (Convergence of SE): Suppose that $\delta > \delta_{\mathrm{AMP}} = \frac{\pi^2}{4} - 1$ *and* $\sigma_w^2 = 0$*. For any* $\alpha_0 \in \mathbb{R}\backslash 0$ *and* $\sigma_0^2 < \infty$*, the sequences* $\{\alpha_t\}_{t\geq1}$ *and* $\{\sigma_t^2\}_{t\geq1}$ *defined in* (III.2) *converge:*

$$\lim_{t\to\infty} |\alpha_t| = 1 \quad \text{and} \quad \lim_{t\to\infty} \sigma_t^2 = 0.$$

Note that in Theorem 5 the sequences converge for any $\sigma_0^2 < \infty$. This result is stronger than the complex-valued counterpart, which requires $0 < |\alpha_0| \leq 1$ and $\sigma_0^2 \leq 1$ (see Theorem 2).

Theorem 6 (Local Convergence of SE): For the noiseless setting where $\sigma_w^2 = 0$*,* $(\alpha, \sigma^2) = (1, 0)$ *is a fixed point of the SE in* (II.2)*. Furthermore, if* $\delta > \delta_{\mathrm{global}}$*, then there exist two constants* $\epsilon_1 > 0$ *and* $\epsilon_2 > 0$ *such that the SE converges to this fixed point for any* $\alpha_0 \in (1 - \epsilon_1, 1)$ *and* $\sigma_0^2 \in (0, \epsilon_2)$*. On the other hand if* $\delta < \delta_{\mathrm{global}}$*, then the SE cannot converge to* $(1, 0)$ *except when initialized there.*

Note that δ_{global} here is different from the information theoretic limit $\delta = 1$. We should emphasize that if we had not used the continuation discussed in (I.4), then the basin of attraction of $(\alpha, \sigma) = (1, 0)$ would be non-empty as long as $\delta > 1$.

Finally, we discuss the performance of AMP.A in the high SNR regime. See [26, Appendix E] for its proof.

Theorem 7 (Noise Sensitivity): Suppose that $\delta > \delta_{\mathrm{AMP}} = \frac{\pi^2}{4} - 1$ *and* $\alpha_0 \in \mathbb{R}\backslash 0$ *and* $\sigma_0^2 < \infty$*. Then, in the high SNR regime we have*

$$\lim_{\sigma_w^2 \to 0} \frac{\mathrm{AMSE}(\sigma_w^2, \delta)}{\sigma_w^2} = \frac{1}{\left(1 + \frac{4}{\pi^2}\right)^{-1} - \frac{1}{\delta}}.$$

IV. Discussions and Future Research

A. Simulation Results of AMP.A

In Sections II and III, we analyzed the performance of AMP.A under the asymptotic setting $m, n \to \infty$, $m/n \to \delta \in (0, \infty)$. For empirical performance of AMP.A and comparisons to existing algorithms, the reader is referred to the conference version of this paper [48].[4] Simulation results in [48] confirmed that AMP.A achieves state-of-the-art performance for optimization-based (for both intensity-based and amplitude-based loss functions) phase retrieval algorithms. In [48], we also investigated the impact of spectral initialization (based on the carefully tuned nonlinearity proposed in [19]) on the performance of AMP.A. Numerical results in [48] show that, for the complex-valued case, using the

[4]The Matlab codes for the simulation results in [48] are available at https://github.com/junjiema2/AMP-for-amplitude-loss-based-phase-retrieval.

spectral initialization in [19] can only provide marginal gain in terms of improving the reconstruction threshold. Similar finding was also observed for the real-valued setting; although we did not provide simulation results in [48] due to space limitation.

After the first submission of this paper, we noticed a recent ArXiv paper [51] in which a successive and incremental algorithm was proposed (termed IncrePR). Intriguingly, simulation results of [51] (see Fig. 1) show that the reconstruction thresholds of the IncrePR algorithm are very close to $\delta_{\rm AMP}$ for both the complex and real-valued models. It is an interesting future work to investigate whether this is a pure coincidence or there are some fundamental connections between AMP.A and IncrePR.

B. Future Research

As already mentioned in previous sections, we are working on a rigorous proof for combining spectral initialization and AMP.A in a future paper [29]. We also plan to discuss issues related to the discontinuity of $g(p, y)$, and the consequence for state evolution.

There are several other possible directions. First, although we focused on the amplitude-based loss in this paper, it is also possible to study the intensity-based loss (which was proposed in [4]) based on a similar AMP framework. Second, the AMP.A algorithm and the theory developed in this paper relies crucially on the i.i.d. Gaussian assumption on $\boldsymbol{A}$. The results of this paper cannot be directly extended to more practically relevant models (which involve Fourier matrices [52]). The extension of AMP.A to Fourier-based measurements is an interesting future work. To this end, the reader is referred to [53]–[58] for related recent progress.

V. Proof of Theorem 2: Convergence of the SE

The goal of this section is to prove Theorem 2. Since the proof is very long, we summarize the organization of this section to help the reader navigate through the complete proof.

1) Section V-A is a proof sketch. The reader can skip the remaining sections if he or she is only interested in the main idea of our proof. This section contains Lemmas 3-6 with proofs postponed in later sections (except for Lemma 4).
2) Section V-B analyzes the properties of the SE maps ψ_1 and ψ_2. This section contains Lemma 7 and Lemma 8, as well as their proofs.
3) Section V-C analyzes the properties of F_1 and F_2, which are introduced in Section V-A and formally defined in Lemma 7.
4) Section V-D proves Lemma 3. This section contains Lemma 9 and its proof.
5) Section V-E proves Lemma 5 that is introduced in Section V-A.
6) Section V-F proves Lemma 6. This section contains Lemma 10-16 and their proofs.

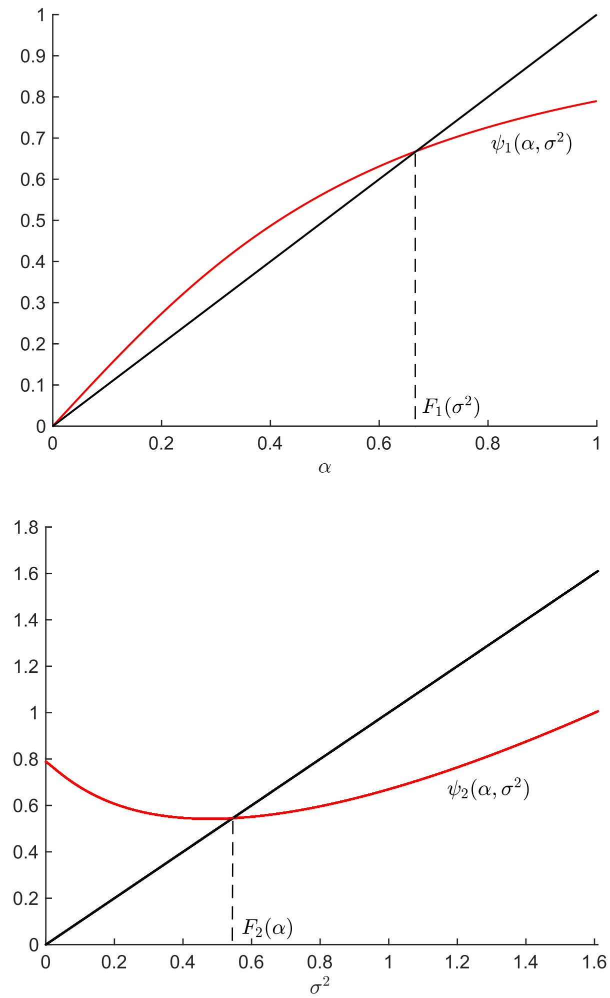

Fig. 2. **Top:** plot of $\psi_1(\alpha, \sigma^2)$ against α. $\sigma^2 = 0.3$. **Bottom:** plot of $\psi_2(\alpha, \sigma^2; \delta)$ against σ^2. $\alpha = 0.3$ and $\delta = \delta_{\rm AMP}$.

A. Roadmap of the Proof

Our main goal is to study the dynamics of the iterations:

$$\begin{aligned} \alpha_{t+1} &= \psi_1(\alpha_t, \sigma_t^2), \\ \sigma_{t+1}^2 &= \psi_2(\alpha_t, \sigma_t^2; \delta). \end{aligned} \tag{V.1}$$

Notice that according to the assumptions of Theorem 2, we assume that we initialized the dynamical system with $\alpha_0 > 0$. Our first hope is that this dynamical system will not oscillate and will converge to the solutions of the following system of nonlinear equations:

$$\begin{aligned} \alpha &= \psi_1(\alpha, \sigma^2), \\ \sigma^2 &= \psi_2(\alpha, \sigma^2; \delta). \end{aligned} \tag{V.2}$$

Hence, the first step is to characterize and understand the fixed points of the solutions of (V.2). Toward this goal, we should study the properties of $\psi_1(\alpha, \sigma^2)$ and $\psi_2(\alpha, \sigma^2; \delta)$. In particular, we would like to know how the fixed points of $\psi_1(\alpha, \sigma^2)$ behave for a given σ^2 and how the fixed points of $\psi_2(\alpha, \sigma^2; \delta)$ behave for a given value of α and δ. The graphs of these functions are shown in Figure 2. We list some of

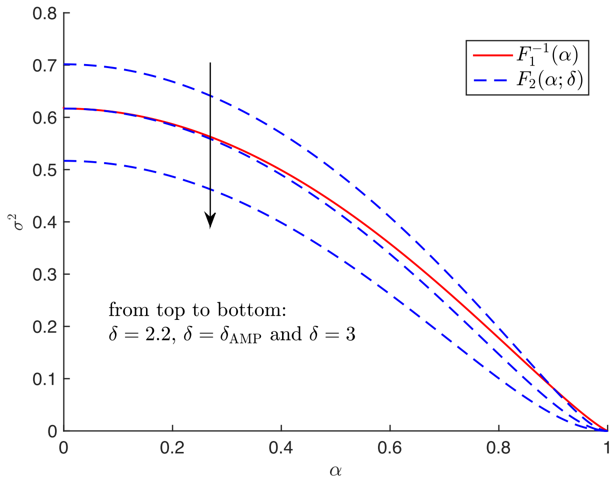

Fig. 3. Plots of $F_1^{-1}(\alpha)$ and $F_2(\alpha;\delta)$ for different values of δ. When $\delta = \delta_{\text{AMP}}$, $F_1^{-1}(\alpha)$ and $F_2(\alpha;\delta)$ intersect at $\alpha = 0$.

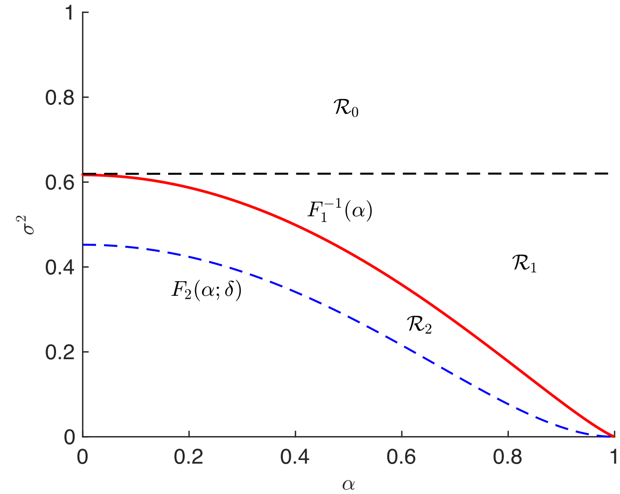

Fig. 4. Illustration of the three regions in Definition 4. Note that $\mathcal{R}_2$ also includes the region below $F_2(\alpha;\delta)$.

the important properties of these two functions. We refer the reader to Section V-B to see more accurate statement of these claims.

1) $\psi_1\left(\alpha,\sigma^2\right)$ is a concave and strictly increasing function of $\alpha > 0$, for any $\sigma^2 > 0$: This implies that $\psi_1\left(\alpha,\sigma^2\right)$ can have two fixed points: one at zero and one at $\alpha > 0$. Also, as is clear from the figure, the second fixed point is the stable one.
2) If $\delta > \delta_{\text{AMP}}$, then ψ_2 has always one stable fixed point. It may have one unstable fixed point (as a function of σ^2). See Fig. 5 for an example of this situation.

For the moment, let us assume that the unstable fixed point does not affect the dynamics of AMP.A. Let $F_1(\sigma^2)$ denote the non-zero fixed point of ψ_1 and $F_2(\alpha;\delta)$ the stable fixed point of ψ_2.[5] We will prove in Lemma 9 that $F_1(\sigma^2)$ is a decreasing function and hence $F_1^{-1}(\alpha)$ is well-defined on $0 < \alpha \leq 1$. Moreover, we will show that by choosing $F_1^{-1}(0) = \frac{\pi^2}{16}$, $F_1^{-1}(\alpha)$ is continuous on [0,1].

$F_1^{-1}(\alpha)$ and $F_2(\alpha;\delta)$ are shown in Fig. 3. Note that the places these curves intersect correspond to the fixed points of (V.2). Depending on the value of δ, the two curves show the following different behaviors:

1) When $\delta > \delta_{\text{AMP}}$, the dashed curve (see Fig. 3) is entirely below the solid curve except at $(\alpha,\sigma^2) = (1,0)$. δ_{AMP} is the critical value of δ at which $F_2(0;\delta) = F_1^{-1}(0)$. Formally, we will prove the following lemma:
Lemma 3: If $\delta \geq \delta_{\text{AMP}} = \frac{64}{\pi^2} - 4$, *then* $F_1^{-1}(\alpha) > F_2(\alpha;\delta)$ *holds for any* $\alpha \in (0,1)$.
The proof of this lemma can be found in Section V-D. Intuitively speaking, in this case we expect the state evolution to converge to the fixed point $(\alpha,\sigma^2) = (1,0)$, meaning that AMP.A achieves exact recovery.
2) When $2 < \delta < \delta_{\text{AMP}}$, the two curves intersect at multiple locations, but $F_2(\alpha;\delta) < F_1^{-1}(\alpha)$ for the values of α that are close to one. This implies that AMP.A can still exactly recover $\boldsymbol{x}_*$ if the initialization is close enough to $\boldsymbol{x}_*$. However, this does not happen with spectral initialization [48]. We will discuss this case in Theorem 3.

So far, we have studied the solutions of (V.2). However, our ultimate goal is to analyze the dynamical behavior of (V.1). In particular, we are interested to see under what conditions will the estimates (α_t,σ_t^2) converge to $(1,0)$ (and do not oscillate). Unfortunately, the dynamics of (α_t,σ_t^2) do not always monotonically move toward the fixed point $(1,0)$, which makes the analysis of SE complicated.

We will first consider the case $\delta > \delta_{\text{AMP}}$. The following lemma shows that (α_t,σ_t^2) lies within a bounded region if the initialization falls into that region.

Lemma 4: Suppose that $\alpha_0 > 0$ *and* $\sigma_0^2 \leq 1$. *If* $\delta > \delta_{\text{AMP}} = \frac{64}{\pi^2} - 4$, *then the sequences* $\{\alpha_t\}_{t\geq 1}$ *and* $\{\sigma_t^2\}_{t\geq 1}$ *generated by* (II.1) *satisfy*

$$0 \leq \alpha_t \leq 1 \quad \text{and } 0 \leq \sigma_t^2 \leq \sigma_{\max}^2, \quad \forall t \geq 1,$$

where $\sigma_{\max}^2 \triangleq \max\left\{1, \frac{4}{\delta}\right\}$.

Proof: As discussed in Lemma 1, the assumption $\alpha_0 > 0$ implies that $\alpha_t > 0$, $\forall t \geq 1$. Further, from the property that $0 < \psi_1(\alpha,\sigma^2) < 1$ for $\alpha > 0$ and $\sigma^2 > 0$ (see Lemma 7 (ii)), we readily have $0 \leq \alpha_t \leq 1$. Similarly, Lemma 8 (iii) shows that if $\delta > \delta_{\text{AMP}}$, $\alpha \in [0,1]$ and $\sigma^2 \in [0,\sigma_{\max}^2]$, then $0 \leq \psi_2(\alpha,\sigma^2;\delta) \leq \sigma_{\max}^2$. From our assumption, we have $\sigma_0^2 \leq 1 \leq \sigma_{\max}^2$. Then, a simple induction argument proves $0 \leq \sigma_t^2 \leq \sigma_{\max}^2$. □

Due to the above lemma, we only need to understand the dynamics of the SE in the region $\mathcal{R} \triangleq \left\{(\alpha,\sigma^2)\middle| 0 < \alpha \leq 1, 0 < \sigma^2 \leq \sigma_{\max}^2\right\}$. Since the dynamic of AMP.A is complicated, we further divide this region into smaller regions. See Fig. 4 for an illustration.

[5] In the literature of dynamical systems, these functions are sometimes called *nullclines*. Nullclines are useful for qualitatively analyzing local dynamical behavior of two-dimensional maps (which is the case for the SE in the present paper).

Definition 4: *We divide* $\mathcal{R} \triangleq \left\{(\alpha, \sigma^2)\middle|0 < \alpha \le 1, 0 < \sigma^2 \le \sigma^2_{\max}\right\}$ *into the following three sub-regions:*

$$\mathcal{R}_0 \triangleq \left\{(\alpha, \sigma^2)\middle|0 < \alpha \le 1, \frac{\pi^2}{16} < \sigma^2 \le \sigma^2_{\max}\right\},$$
$$\mathcal{R}_1 \triangleq \left\{(\alpha, \sigma^2)\middle|0 < \alpha \le 1, F_1^{-1}(\alpha) < \sigma^2 \le \frac{\pi^2}{16}\right\},$$
$$\mathcal{R}_2 \triangleq \left\{(\alpha, \sigma^2)\middle|0 < \alpha \le 1, 0 \le \sigma^2 \le F_1^{-1}(\alpha)\right\}. \quad \text{(V.3)}$$

Our next lemma shows that if (α_t, σ_t^2) is in $\mathcal{R}_1$ or $\mathcal{R}_2$ for $t \ge 1$, then (α_t, σ_t^2) converges to the desired fixed point $(1, 0)$. In other words, if we initialize the SE in $\mathcal{R}_1$ or $\mathcal{R}_2$, then after only one iteration, the estimates will lie inside of the "attraction basin" of $(1, 0)$.

Lemma 5: *Suppose that* $\delta > \delta_{\text{AMP}}$. *If* $(\alpha_{t_0}, \sigma^2_{t_0})$ *is in* $\mathcal{R}_1 \cup \mathcal{R}_2$ *at time* t_0 *(where* $t_0 \ge 1$*), and* $\{\alpha_t\}_{t\ge t_0}$ *and* $\{\sigma_t^2\}_{t\ge t_0}$ *are obtained via the SE in* (II.1)*, then*

(i) (α_t, σ_t^2) *remains in* $\mathcal{R}_1 \cup \mathcal{R}_2$ *for all* $t > t_0$*;*

(ii) (α_t, σ_t^2) *converges:*

$$\lim_{t\to\infty} \alpha_t = 1 \quad \text{and} \quad \lim_{t\to\infty} \sigma_t^2 = 0.$$

This claim will be proved in Section V-E. Notice that the condition $t_0 \ge 1$ is important for part (i) to hold: if (α_0, σ_0^2) is close to the origin (and thus in $\mathcal{R}_2$), then (α_1, σ_1^2) can move to $\mathcal{R}_0$. However, this cannot happen when $t \ge 1$. In the proof given in Section V-E, we showed that for any $(\alpha_0, \sigma_0^2) \in \mathcal{R}$ the possible locations of (α_1, σ_1^2) are bounded from below by a curve, and once (α, σ^2) is above this curve and also in region $\mathcal{R}_1$ or $\mathcal{R}_2$, then it cannot go to $\mathcal{R}_0$. The following lemma, together with Lemma 5, completes the proof of Theorem 2.

Lemma 6: *Suppose that* $\delta > \delta_{\text{AMP}}$. *Let* $\{\alpha_t\}_{t\ge1}$ *and* $\{\sigma_t^2\}_{t\ge1}$ *be the sequences generated according to* (II.1) *from any* $(\alpha_0, \sigma_0^2) \in \mathcal{R}_0$. *Then, there exists a finite number* $T \ge 1$ *such that* $(\alpha_T, \sigma_T^2) \in \mathcal{R}_1 \cup \mathcal{R}_2$.

The proof of this result is detailed in Section V-F. Combining the above two lemmas, it is straightforward to see that $(\alpha_t, \sigma_t^2) \to (1, 0)$, and hence the proof is complete.

In the following subsections, we present the details that are missing in the above proof sketch.

B. Properties of ψ_1 and ψ_2

In this section we derive all the main properties of ψ_1 and ψ_2 that are used throughout the paper.

Lemma 7: $\psi_1\left(\alpha, \sigma^2\right)$ *has the following properties (for* $\alpha \ge 0$*):*

(i) $\psi_1\left(\alpha, \sigma^2\right)$ *is a concave and strictly increasing function of* $\alpha > 0$*, for any given* $\sigma^2 > 0$.

(ii) $0 < \psi_1(\alpha, \sigma^2) \le 1$*, for* $\alpha > 0$ *and* $\sigma^2 > 0$.

(iii) *If* $0 < \sigma^2 < \frac{\pi^2}{16}$*, then there are two nonnegative solutions to* $\alpha = \psi_1(\alpha, \sigma^2)$*:* $\alpha = 0$ *and* $\alpha = F_1(\sigma^2) > 0$. *Further,* $F_1(\sigma^2)$ *is strongly globally attracting, meaning that*

$$\alpha < \psi_1(\alpha, \sigma^2) < F_1(\sigma^2), \quad \alpha \in (0, F_1(\sigma^2)), \quad \text{(V.4a)}$$

and

$$F_1(\sigma^2) < \psi_1(\alpha, \sigma^2) < \alpha, \quad \alpha \in (F_1(\sigma^2), \infty). \quad \text{(V.4b)}$$

On the other hand, if $\sigma^2 \ge \frac{\pi^2}{16}$*, then* $\alpha = 0$ *is the unique nonnegative fixed point and it is strongly globally attracting.*

Proof:

Part (i): From (II.2), it is easy to verify that $\psi_1(\alpha, \sigma^2)$ is an increasing function of $\alpha > 0$. We now prove its concavity. To this end, we calculate its first and second partial derivatives ($\forall \alpha > 0, \sigma^2 > 0$):

$$\frac{\partial \psi_1(\alpha, \sigma^2)}{\partial \alpha} = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cdot \sigma^2}{(\alpha^2 \sin^2\theta + \sigma^2)^{\frac{3}{2}}} \mathrm{d}\theta, \quad \text{(V.5a)}$$
$$\frac{\partial_1^2 \psi_1(\alpha, \sigma^2)}{\partial \alpha^2} = \int_0^{\frac{\pi}{2}} \frac{-3\sin^4\theta \cdot \sigma^2 \alpha}{(\alpha^2 \sin^2\theta + \sigma^2)^{\frac{5}{2}}} \mathrm{d}\theta < 0, \quad \text{(V.5b)}$$

Hence, $\psi_2(\alpha, \sigma^2)$ is a concave function of α for $\alpha > 0$.

Part (ii): Positivity of ψ_1 is obvious. Also, note that

$$\psi_1(\alpha, \sigma^2) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(\sin^2(\theta) + \frac{\sigma^2}{\alpha^2})^{\frac{1}{2}}} d\theta \le \int_0^{\frac{\pi}{2}} \sin\theta d\theta = 1.$$

Part (iii): The claim is a consequence of the concavity of ψ_1 (with respect to α) and the following condition:

$$\left.\frac{\partial \psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha=0} = 1 \Longleftrightarrow \sigma^2 = \frac{\pi^2}{16}.$$

The detailed proof is as follows. First, it is straightforward to verify that $\alpha = 0$ is always a solution to $\alpha = \psi_1(\alpha, \sigma^2)$. Define

$$\Psi_1(\alpha, \sigma^2) \triangleq \psi_1(\alpha, \sigma^2) - \alpha.$$

Since $\Psi_1(\alpha, \sigma^2)$ is a concave function of α (as $\psi_1(\alpha, \sigma^2)$ is concave), $\frac{\partial \Psi_1(\alpha, \sigma^2)}{\partial \alpha}$ is decreasing. Let's first consider $\sigma^2 > \frac{\pi^2}{16}$. In this case we know that

$$\begin{aligned} \frac{\partial \Psi_1(\alpha, \sigma^2)}{\partial \alpha} &\le \left.\frac{\partial \Psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha=0} \\ &= \left.\frac{\partial \psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha=0} - 1 \\ &= \frac{\pi}{4\sigma} - 1 < 0, \end{aligned} \quad \text{(V.6)}$$

where the second equality can be calculated from (V.5a). Since $\Psi_1(\alpha, \sigma^2)$ is a decreasing function of α and is equal to zero when $\alpha = 0$, it follows that $\Psi_1(\alpha, \sigma^2) = 0$ does not have any other solution except for $\alpha = 0$. Now, consider case $\sigma^2 < \frac{\pi^2}{16}$. It is straightforward to confirm that

$$\left.\frac{\partial \Psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha=0} = \left.\frac{\partial \psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha=0} - 1 = \frac{\pi}{4\sigma} - 1 > 0.$$

Furthermore, from (V.5a) we have $\left.\frac{\partial \psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha\to\infty} = 0$, and so

$$\left.\frac{\partial \Psi_1(\alpha, \sigma^2)}{\partial \alpha}\right|_{\alpha\to\infty} \to -1.$$

Hence, $\Psi_1(\alpha, \sigma^2) = 0$ has exactly one more solution for $\alpha > 0$. Note that since from part (ii) $\psi_1(\alpha, \sigma^2) < 1$, the solution of $\alpha = \psi_1(\alpha, \sigma^2)$ also satisfies $\alpha \le 1$.

Finally, the strong global attractiveness follows from the fact that ψ_1 is a strictly increasing function of α. □

Lemma 8: $\psi_2\left(\alpha,\sigma^2;\delta\right)$ *has the following properties:*

(i) *If* $\delta < 2$, *then* $\sigma^2 = 0$ *is a locally unstable fixed point to* $\sigma^2 = \psi_2\left(\alpha,\sigma^2;\delta\right)$, *meaning that*

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}\Big|_{\alpha=1,\sigma^2=0} > 1.$$

(ii) *For any* $\delta > 2$, $\sigma^2 = \psi_2\left(\alpha,\sigma^2;\delta\right)$ *has a unique fixed point, denoted as* $F_2(\alpha;\delta)$, *in* $\sigma^2\in[0,1]$ *for any* $\alpha\in[0,1]$. *Further,* $F_2(\alpha;\delta)$ *is (weakly) globally attracting in* $\sigma^2\in[0,1]$*:*

$$\sigma^2 < \psi_2(\alpha,\sigma^2;\delta),\quad \sigma^2\in(0,F_2(\alpha;\delta)),\tag{V.7a}$$

and

$$\psi_2(\alpha,\sigma^2) < \sigma^2,\quad \sigma^2\in(F_2(\alpha;\delta),1).\tag{V.7b}$$

(iii) *If* $\delta\geq\delta_{\text{AMP}}$, *then for any* $\alpha\in[0,1]$, *we have*

$$0\leq\psi_2(\alpha,\sigma^2;\delta)\leq\sigma^2_{\max},\quad \sigma^2\in[0,\sigma^2_{\max}],$$

where $\sigma^2_{\max}\triangleq\max\{1,4/\delta\}$.

(iv) *If* $\delta\geq\delta_{\text{AMP}}$, *then for any* $\alpha\in[0,1]$, $F_2(\alpha;\delta)$ *is the unique (weakly) globally attracting fixed point of* $\sigma^2 = \psi_2(\alpha,\sigma^2;\delta)$ *in* $\sigma^2\in[0,\sigma^2_{\max}]$. *Namely,*

$$\sigma^2 < \psi_2(\alpha,\sigma^2;\delta),\quad \sigma^2\in(0,F_2(\alpha;\delta)),\tag{V.8a}$$

and

$$\psi_2(\alpha,\sigma^2) < \sigma^2,\quad \sigma^2\in(F_2(\alpha;\delta),\sigma^2_{\max}).\tag{V.8b}$$

(v) *For any* $\delta > 0$, $\psi_2(\alpha,\sigma^2;\delta)$ *is an increasing function of* $\sigma^2 > 0$ *if*

$$\alpha > \alpha_*\triangleq\frac{1}{2\sqrt{1+s_*^2}}E\left(\frac{1}{1+s_*^2}\right)\approx 0.53,\tag{V.9}$$

where s_*^2 *is the unique solution to*

$$2E\left(\frac{1}{1+s_*^2}\right) = K\left(\frac{1}{1+s_*^2}\right).$$

Here, $K(\cdot)$ *and* $E(\cdot)$ *denote the complete elliptic integrals introduced in* (A.1). *Further, when* $\alpha > \alpha_*$ *and* $\delta > \delta_{\text{AMP}}$, *then* $F_2(\sigma^2;\delta)$ *is strongly globally attracting in* $[0,\sigma^2_{\max}]$. *Specifically,*

$$\sigma^2 < \psi_2(\alpha,\sigma^2;\delta) < F_2(\alpha;\delta),\quad \sigma^2\in(0,F_2(\alpha;\delta)),$$

and

$$F_2(\alpha;\delta) < \psi_2(\alpha,\sigma^2) < \sigma^2,\quad \sigma^2\in(F_2(\alpha;\delta),\sigma^2_{\max}).$$

Proof: First note that the partial derivative of ψ_2 w.r.t. σ^2 is given by

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} = \frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sigma^2}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{3}{2}}}\mathrm{d}\theta\right).\tag{V.10}$$

Part (i): Before we proceed, we first comment on the discontinuity of the partial derivative $\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}$ at $\sigma^2=0$. Note that the formula in (V.10) was derived for non-zero values of σ^2. Naively, one may plug in $\sigma^2=0$ in the equation and assume that $\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}\Big|_{\alpha=1,\sigma^2=0} = \frac{4}{\delta}$. This is not the case since the integral $\int_0^{\pi/2}\frac{d\theta}{\sin\theta}$ is divergent. It turns out that the derivative $\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}$ is a continuous function of σ^2. The technical details can be found in [26, Appendix C].

Since $\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}$ is continuous at $\sigma^2=0$, we have

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}\Big|_{\alpha=1,\sigma^2=0} = \lim_{\sigma^2\to0}\frac{\partial\psi_2(1,\sigma^2;\delta)}{\partial\sigma^2}.$$

Note that if we set $m=1/\sigma^2$, then from (A.5) we have

$$\begin{aligned}\frac{\partial\psi_2(1,\sigma^2;\delta)}{\partial\sigma^2} &= \frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sigma^2}{(\sin^2\theta+\sigma^2)^{\frac{3}{2}}}\mathrm{d}\theta\right)\\ &= \frac{4}{\delta}\left(1-\frac{1}{2}\sqrt{\frac{m}{1+m}}E\left(\frac{m}{m+1}\right)\right).\end{aligned}$$

It is then straightforward to use Lemma 18 to prove that

$$\lim_{m\to\infty}\frac{4}{\delta}\left(1-\frac{1}{2}\sqrt{\frac{m}{1+m}}E\left(\frac{m}{m+1}\right)\right) = \frac{2}{\delta}.$$

Hence, $\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}\Big|_{\alpha=1,\sigma^2=0} > 1$ for $\delta < 2$.

Part (ii): We first prove that the following equation has at least one solution for any $\alpha\in[0,1]$ and $\delta>2$:

$$\sigma^2 = \psi_2(\alpha,\sigma^2;\delta),\quad \sigma^2\in[0,1].$$

It is straightforward to verify that

$$\psi_2(\alpha,\sigma^2;\delta)|_{\sigma^2=0} = \frac{4}{\delta}(1-\alpha)^2\geq 0.\tag{V.11}$$

We next prove our claim by proving

$$\psi_2(\alpha,\sigma^2;\delta)|_{\sigma^2=1} < 1,\quad \forall\alpha\in[0,1]\text{ and }\delta>2.\tag{V.12}$$

From (II.2c), we have

$$\psi_2(\alpha,\sigma^2;\delta)|_{\sigma^2=1} < 1 \iff \underbrace{\int_0^{\frac{\pi}{2}}\frac{2\alpha^2\sin^2\theta+1}{(\alpha^2\sin^2\theta+1)^{\frac{1}{2}}}\mathrm{d}\theta-\alpha^2}_{g(\alpha^2)} > 2-\frac{\delta}{4}.\tag{V.13}$$

We next show that $g(\alpha^2)$ in (V.13) is a concave function of α^2. The first two derivatives w.r.t. α^2 are given by:

$$\frac{\mathrm{d}g(\alpha^2)}{\mathrm{d}\alpha^2} = \int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\left(\alpha^2\sin^2\theta+\frac{3}{2}\right)}{(\alpha^2\sin^2\theta+1)^{\frac{3}{2}}}\mathrm{d}\theta-1,$$

and

$$\frac{\mathrm{d}^2g(\alpha^2)}{\mathrm{d}(\alpha^2)^2} = -\int_0^{\frac{\pi}{2}}\frac{\sin^4\theta\left(\frac{1}{2}\alpha^2\sin^2\theta+\frac{5}{4}\right)}{(\alpha^2\sin^2\theta+1)^{\frac{5}{2}}}\mathrm{d}\theta < 0.$$

The concavity of $g(\alpha^2)$ implies that its minimum happens at either $\alpha=0$ or $\alpha=1$. Hence, to prove (V.13), it suffices to prove that

$$g(0)=\frac{\pi}{2} > 2-\frac{\delta}{4}\quad\text{and}\quad g(1)\approx 1.509 > 2-\frac{\delta}{4},$$

which holds for $\delta>2$. Hence, (V.13) holds. By combining (V.11) and (V.12) we conclude that $\psi_2(\alpha,\sigma^2;\delta)$ has at least one fixed point between $\sigma^2=0$ and $\sigma^2=1$. The next step is to prove the uniqueness of this fixed point. For the rest of the proof, we discuss two cases separately: a) $\delta > 4$ and b) $2 < \delta \le 4$.

(a) $\delta > 4$. Define

$$\Psi_2(\alpha, \sigma^2; \delta) \triangleq \psi_2(\alpha, \sigma^2; \delta) - \sigma^2. \tag{V.14}$$

From (V.10), if $\delta > 4$, then $\frac{\partial \psi_2(\alpha,\sigma^2;\delta)}{\partial \sigma^2} < 1$, $\forall \sigma^2 > 0$. This means that $\Psi_2(\alpha, \sigma^2; \delta)$ defined in (V.14) is monotonically decreasing in $\sigma^2 > 0$. Hence, the solution to $\Psi_2(\alpha, \sigma^2; \delta) = 0$ is unique. Furthermore, the following property is a direct consequence of the monotonicity of $\Psi_2(\alpha, \sigma^2; \delta)$:

$$\Psi_2(\alpha, \sigma^2; \delta) < 0, \quad \forall 0 < \sigma^2 < F_2(\alpha; \delta), \tag{V.15a}$$

and

$$\Psi_2(\alpha, \sigma^2; \delta) > 0 > \sigma^2, \quad \forall F_2(\alpha; \delta) < \sigma^2 < 1, \tag{V.15b}$$

where $F_2(\alpha; \delta)$ denotes the solution to $\Psi_2(\alpha, \sigma^2; \delta) = 0$.

(b) $2 < \delta \le 4$. In this case, we will prove that there exists a threshold on σ^2, denoted as $\sigma_\star^2(\alpha; \delta)$ below, such that the following hold:

$$\frac{\partial \psi_2(\alpha, \sigma^2; \delta)}{\partial \sigma^2} < 1, \quad \forall \sigma^2 < \sigma_\star^2(\alpha; \delta) \tag{V.16a}$$

and

$$\frac{\partial \psi_2(\alpha, \sigma^2; \delta)}{\partial \sigma^2} > 1, \quad \forall \sigma^2 \in (\sigma_\star^2(\alpha; \delta), \infty). \tag{V.16b}$$

This means that $\Psi_2(\alpha, \sigma^2; \delta) = \psi_2(\alpha, \sigma^2; \delta) - \sigma^2$ is strictly decreasing on $\sigma^2 \in (0, \sigma_\star^2(\alpha; \delta))$ and increasing on $\sigma^2 \in (\sigma_\star^2(\alpha; \delta), \infty)$. Note that since we have proved that $\Psi_2(\alpha, \sigma^2; \delta) = 0$ has at least one solution, we conclude that there exist exactly two solutions to $\Psi_2(\alpha, \sigma^2; \delta) = 0$, one in $(0, \sigma_\star^2(\alpha; \delta))$ and the second in $(\sigma_\star^2(\alpha; \delta), \infty)$, if $\Psi_2(\alpha, \sigma^2; \delta)|_{\sigma^2=\sigma_\star^2(\alpha;\delta)} < 0$. This is the case since $\Psi_2(\alpha, \sigma^2; \delta)|_{\sigma^2=1} < 0$ (see (V.12)), and that $\Psi_2(\alpha, \sigma^2; \delta)|_{\sigma^2=1} < \Psi_2(\alpha, \sigma^2; \delta)|_{\sigma^2=\sigma_\star^2(\alpha;\delta)}$ (since the latter is the global minimum of $\Psi_2(\alpha, \sigma^2; \delta)$ in $\sigma^2 \in (0, \infty)$).

Also, it is easy to prove (V.15). In fact, the following holds:

$$\Psi_2(\alpha, \sigma^2; \delta) < 0, \quad \forall 0 < \sigma^2 < F_2(\alpha; \delta),$$

and

$$\Psi_2(\alpha, \sigma^2; \delta) > 0 > \sigma^2, \quad \forall F_2(\alpha; \delta) < \sigma^2 < \hat{F}_2(\alpha; \delta),$$

where $\hat{F}_2(\alpha; \delta) > 1$ denotes the larger solution to $\Psi_2(\alpha, \sigma^2; \delta) = 0$. See Fig. 5 for an illustration.

From the above discussions, it remains to prove (V.16). To this end, it is more convenient to express (V.10) using elliptic integrals defined in Section VI-B:

$$\frac{\partial \psi_2(\alpha, \sigma^2; \delta)}{\partial \sigma^2} = \frac{4}{\delta}\left(1 - \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sigma^2}{(\alpha^2 \sin^2\theta + \sigma^2)^{\frac{3}{2}}} d\theta\right) \tag{V.17a}$$

$$= \frac{4}{\delta\alpha}\Bigg(\alpha - \underbrace{\frac{1}{2\sqrt{1+s^2}} E\left(\frac{1}{1+s^2}\right)}_{f(s)}\Bigg), \tag{V.17b}$$

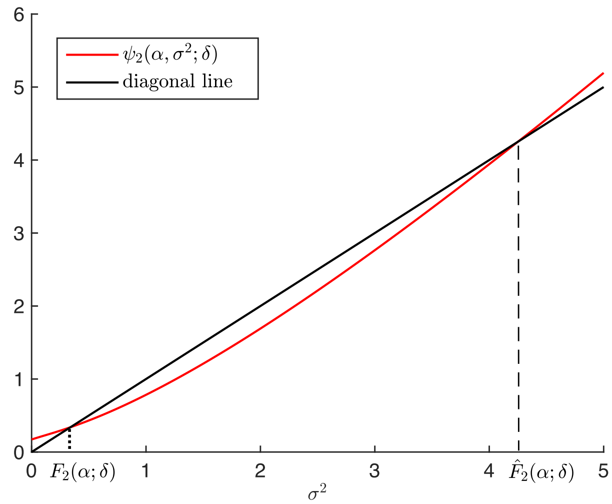

Fig. 5. Plot of $\psi_2(\alpha, \sigma^2; \delta)$ for $\alpha = 0.7$ and $\delta = 2.1$.

where we introduced a new variable $s \triangleq \frac{\sigma}{\alpha}$ and the last step is derived using the identities in Lemma 19. Based on (V.17) we can now rewrite (V.16) as

$$f(s) > \alpha\left(1 - \frac{\delta}{4}\right), \quad \forall s < \frac{\sigma_\star(\alpha; \delta)}{\alpha} \tag{V.18a}$$

and

$$f(s) < \alpha\left(1 - \frac{\delta}{4}\right), \quad \forall s \in \left(\frac{\sigma_\star(\alpha; \delta)}{\alpha}, \infty\right). \tag{V.18b}$$

To prove this, we first show that there exists s^* such that $f(s)$ is strictly increasing on $(0, s_*)$ and decreasing on (s_*, ∞), namely,

$$f'(s) > 0, \text{ for } s < s_*, \quad \text{and } f'(s) < 0, \text{ for } s > s_*. \tag{V.19a}$$

s_* is in fact the unique solution to the following equation:

$$2E\left(\frac{1}{1+s_*^2}\right) = K\left(\frac{1}{1+s_*^2}\right). \tag{V.19b}$$

This can be seen from $f'(s)$ derived below:

$$f'(s) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2\sqrt{1+s^2}} E\left(\frac{1}{1+s^2}\right)$$
$$= \frac{s}{2(1+s^2)^{\frac{3}{2}}}\left[K\left(\frac{1}{1+s^2}\right) - 2E\left(\frac{1}{1+s^2}\right)\right].$$

Further noting that $E(\cdot)$ is strictly decreasing in $(0, 1)$ while $K(\cdot)$ is increasing, we proved (V.19).

Based on the above discussions, we can finally turn to the proof of (V.18). From (V.17b), it is straightforward to verify that $f(0) = \frac{1}{2}$. Therefore, when $\delta > 2$, we have

$$\alpha\left(1 - \frac{\delta}{4}\right) \le 1 - \frac{\delta}{4} < \frac{1}{2} = f(0), \quad \forall \delta > 2 \text{ and } 0 \le \alpha \le 1.$$

Hence, the following equation admits a unique solution (denoted as $s_\star(\alpha; \delta)$ below):

$$f(s) = \alpha\left(1 - \frac{\delta}{4}\right), \quad \forall \delta > 2 \text{ and } 0 \le \alpha \le 1.$$

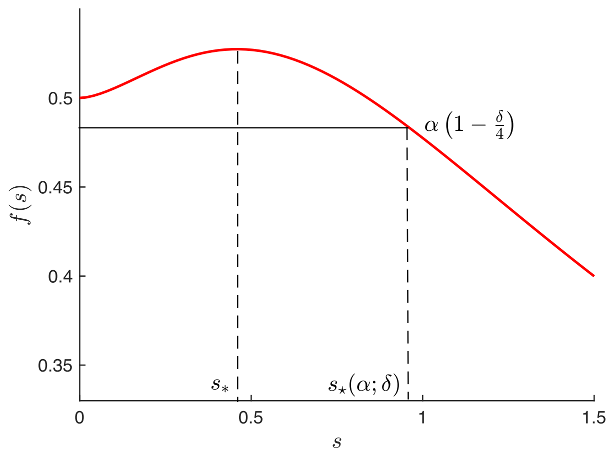

Fig. 6. A plot of $f(s)$.

See Fig. 6 for an illustration. Also, from our above discussions on the monotonicity of $f(s)$ it is straightforward to show that

$$f(s) > \alpha\left(1-\frac{\delta}{4}\right),\quad \forall s < s_\star(\alpha;\delta)$$

and

$$f(s) < \alpha\left(1-\frac{\delta}{4}\right),\quad \forall s \in (s_\star(\alpha;\delta),\infty),$$

which proves (V.18) by setting $\sigma_\star(\alpha;\delta) \triangleq \alpha\cdot s_\star(\alpha;\delta)$. This proves (V.16), which completes the proof.

Part (iii): We will prove a stronger result: $\psi_2 \le 4/\delta$. From (II.2c), $\psi_2(\alpha,\sigma^2;\delta) \le 4/\delta$ is equivalent to

$$\alpha^2+\sigma^2-\int_0^{\frac{\pi}{2}}\frac{2\alpha^2\sin^2\theta+\sigma^2}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta \le 0,$$

which can be further reformulated as

$$\alpha^2 \le \int_0^{\frac{\pi}{2}}\frac{2\alpha^2\sin^2\theta}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta + \sigma^2\left(\int_0^{\frac{\pi}{2}}\frac{1}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta-1\right). \quad \text{(V.20)}$$

For $0\le\alpha\le1$ and $\sigma^2\le\sigma^2_{\max}$ we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}&\frac{1}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta\\ &\ge \int_0^{\frac{\pi}{2}}\frac{1}{\left(\sin^2\theta+\sigma^2_{\max}\right)^{\frac{1}{2}}}\mathrm{d}\theta,\\ &\overset{(a)}{=} \int_0^{\frac{\pi}{2}}\frac{1}{\left(\sin^2\theta+\frac{4}{\delta_{\mathrm{AMP}}}\right)^{\frac{1}{2}}}\mathrm{d}\theta\\ &\approx 1.09 > 1,\end{aligned} \quad \text{(V.21)}$$

where step (a) from $\sigma^2_{\max} = \max\{1,4/\delta\} \ge \max\{1,4/\delta_{\mathrm{AMP}}\} = 4/\delta_{\mathrm{AMP}} \approx 1.6$. Due to (V.21), to prove (V.20), it suffices to prove

$$\alpha^2 \le \int_0^{\frac{\pi}{2}}\frac{2\alpha^2\sin^2\theta}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta,$$

or

$$1 \le \int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta,$$

which, similar to (V.21), can be proved by the following inequality

$$\int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta \ge \int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta}{\left(\sin^2\theta+\frac{4}{\delta_{\mathrm{AMP}}}\right)^{\frac{1}{2}}}\mathrm{d}\theta \approx 1.02 > 1.$$

Part (iv): We bound the partial derivative of $\psi_2(\alpha,\sigma^2;\delta)$ for $\sigma^2\in[0,\sigma^2_{\max}]$ as:

$$\begin{aligned}\frac{\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} &= \frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sigma^2}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{3}{2}}}\mathrm{d}\theta\right)\\ &\overset{(a)}{\le} \frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sigma^2}{(\theta^2+\sigma^2)^{\frac{3}{2}}}\mathrm{d}\theta\right)\\ &\overset{(b)}{=} \frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2\sigma}}\frac{1}{(\tilde{\theta}^2+1)^{\frac{3}{2}}}\mathrm{d}\tilde{\theta}\right)\\ &\overset{(c)}{\le} \frac{4}{\delta_{\mathrm{AMP}}}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2\sqrt{\frac{4}{\delta_{\mathrm{AMP}}}}}}\frac{1}{(\tilde{\theta}^2+1)^{\frac{3}{2}}}\mathrm{d}\tilde{\theta}\right)\\ &\approx 0.98 < 1,\end{aligned} \quad \text{(V.22)}$$

where step (a) follows from the constraint $0\le\alpha\le1$ and the inequality $\sin\theta\le\theta$; (b) is due to the variable change $\tilde{\theta}=\theta/\sigma$; (c) is a consequence of the constraint $\sigma^2\le\sigma^2_{\max} = \max\{1,4/\delta\} \le \max\{1,4/\delta_{\mathrm{AMP}}\} = 4/\delta_{\mathrm{AMP}}$. As a result of (V.22), $\Psi_2(\alpha,\sigma^2;\delta)=\psi_2(\alpha,\sigma^2;\delta)-\sigma^2$ is decreasing in $\sigma^2\in[0,\sigma^2_{\max}]$. It is easy to verify that $\psi_2(0,\alpha;\delta)\ge0$ for $\alpha\in[0,1]$. Further, Lemma 8 (iii) implies that

$$\psi_2(\sigma^2_{\max},\alpha;\delta)-\sigma^2_{\max}\le 0.$$

Hence, there exists a unique solution (which we denote as $F_2(\alpha;\delta)$) to the following equation:

$$\psi_2(\sigma,\alpha;\delta)=\sigma^2,\quad 0\le\sigma^2\le\sigma^2_{\max}.$$

Finally, the property in (V.8) is a direct consequence of the fact that $\Psi_2(\alpha,\sigma^2;\delta)=\psi_2(\alpha,\sigma^2;\delta)-\sigma^2$ is a decreasing function of $\sigma^2\le\sigma^2_{\max}$.

Part (v): In (V.17), we have derived the following:

$$\frac{\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}=\frac{4}{\delta\alpha}\left(\alpha-f(s)\right),$$

where $s\triangleq\frac{\sigma}{\alpha}$. From (V.17b), we see that $\psi_2(\alpha,\sigma^2;\delta)$ is an increasing function of σ^2 if the following holds:

$$\alpha > f(s).$$

Further, (V.19) implies that the maximum of $f(s)$ happens at s_*, i.e.,

$$\max_{s>0}\ f(s)=\frac{1}{2\sqrt{1+s_*^2}}E\left(\frac{1}{1+s_*^2}\right)\triangleq\alpha_*, \quad \text{(V.23)}$$

where s_*^2 is the unique solution to

$$2E\left(\frac{1}{1+s_*^2}\right) = K\left(\frac{1}{1+s_*^2}\right). \tag{V.24}$$

Clearly, $\alpha > \alpha_*$ immediately implies $\alpha > f(s)$, which further guarantees that $\psi_2(\alpha, \sigma^2; \delta)$ is monotonically increasing on $\sigma^2 > 0$. Finally, the strong global attractiveness of $F_2(\alpha; \delta)$ is a direct consequence of part (iv) of this lemma together with the monotonicity of ψ_2. □

C. Properties of F_1 and F_2

In this section we derive the main properties of the functions F_1 and F_2 introduced in Section V-A. These properties play major roles in the results of the paper.

Lemma 9: The following hold for $F_1(\sigma^2)$ and $F_2(\alpha; \delta)$ (for $\delta > 2$):

(i) *$F_1(0) = 1$ and $\lim_{\sigma^2 \to \frac{\pi^2}{16}^-} F_1(\sigma^2) = 0$. Further, by choosing $F_1(\frac{\pi^2}{16}) = 0$, we have $F_1(\sigma^2)$ is continuous on $\left[0, \frac{\pi^2}{16}\right]$ and strictly decreasing in $\left(0, \frac{\pi^2}{16}\right)$;*

(ii) *F_2 is a continuous function of $\alpha \in [0, 1]$ and $\delta \in (2, \infty)$. $F_2(1; \delta) = 0$, and $F_2(0; \delta) = \left(\frac{-\pi+\sqrt{\pi^2+4(\delta-4)}}{\delta-4}\right)^2$ for $\delta \neq 4$ and $F_2(0; 4) = \frac{4}{\pi^2}$.*

Proof:

Part (i): We first verify $F_1(0) = 1$ and $\lim_{\sigma^2 \to \frac{\pi^2}{16}^-} F_1(\sigma^2) = 0$. First, $F_1(0) = 1$ can be seen from the following facts: (a) $\psi_1(\alpha, 0) = 1$ for $\alpha > 0$, see (II.2a); and (b) By definition, $F_1(0)$ is the non-zero solution to $\alpha = \psi_1(\alpha, 0)$. Then, by Lemma 7 (iii) and continunity of ψ_1, we know F_1 is continuous on $[0, \frac{\pi^2}{16})$, and further $\lim_{\sigma^2 \to \frac{\pi^2}{16}^-} F_1(\sigma^2) = 0$ since $\sigma^2 = \frac{\pi^2}{16}$ corresponds to a case where the non-negative solution to $\psi_1(\alpha, \sigma^2) = \alpha$ decreases to zero. Next, we prove the monotonicity of F_1. Note that

$$F_1(\sigma^2) = \psi_1(F_1(\sigma^2), \sigma^2),$$

Differentiation w.r.t. σ^2 yields

$$F_1'(\sigma^2) = \partial_2\psi_1(F_1(\sigma^2), \sigma^2) + \partial_1\psi_1(F_1(\sigma^2), \sigma^2) \cdot F_1'(\sigma^2),$$

where $\partial_2\psi_1(F_1(\sigma^2), \sigma^2) \triangleq \frac{\partial\psi_1(\alpha,\sigma^2)}{\partial\sigma^2}\Big|_{\alpha=F_1(\sigma^2)}$ and $\partial_1\psi_1(F_1(\sigma^2), \sigma^2) \triangleq \frac{\partial\psi_1(\alpha,\sigma^2)}{\partial\alpha}\Big|_{\alpha=F_1(\sigma^2)}$. Hence,

$$\left[1 - \partial_1\psi_1(F_1(\sigma^2), \sigma^2)\right] \cdot F_1'(\sigma^2) = \partial_2\psi_1(F_1(\sigma^2), \sigma^2). \tag{V.25}$$

We have proved in (V.6) that $\frac{\partial\psi_1(\alpha,\sigma^2)}{\partial\alpha}\Big|_{\alpha=0} < 1$ when $\sigma^2 < \frac{\pi^2}{16}$. Together with the concavity of ψ_1 w.r.t. α (cf. Lemma 7 (i)), we have

$$\frac{\partial\psi_1(\alpha, \sigma^2)}{\partial\alpha}\Big|_{\alpha=F_1(\sigma^2)} < \frac{\partial\psi_1(\alpha, \sigma^2)}{\partial\alpha}\Big|_{\alpha=0} < 1, \ \forall\sigma^2 < \frac{\pi^2}{16}. \tag{V.26}$$

Further, from (II.2a), it is straightforward to see that ψ_1 is a strictly decreasing function of σ^2, and thus

$$\partial_2\psi_1(F_1(\sigma^2), \sigma^2) = \frac{\partial\psi_1(\alpha, \sigma^2)}{\partial\alpha}\Big|_{\alpha=F_1(\sigma^2)} < 0. \tag{V.27}$$

Substituting (V.26) and (V.27) into (V.25), we obtain

$$F_1'(\sigma^2) < 0, \quad \forall\sigma^2 < \frac{\pi^2}{16}.$$

Proof of (ii): By Lemma 8 (ii) and continuity of ψ_2, it is straightforward to check that F_2 is continuous. Moreover, we have proved that $\sigma^2 = F_2(\alpha; \delta)$ is the unique solution to the following equation (for $\delta > 2$ and $\sigma^2 \in [0, 1]$):

$$\sigma^2 = \frac{4}{\delta}\left(\alpha^2 + \sigma^2 + 1 - \int_0^{\frac{\pi}{2}} \frac{2\alpha^2\sin^2\theta + \sigma^2}{(\alpha^2\sin^2\theta + \sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta\right). \tag{V.28}$$

When $\alpha = 0$, (V.28) reduces

$$\sigma^2 = \frac{4}{\delta}\left(\sigma^2 + 1 - \frac{\pi}{2}\sigma\right), \quad \sigma^2 \in [0, 1],$$

which has two possible solutions (for $\delta \neq 4$):

$$\sigma_1 = \frac{-\pi + \sqrt{\pi^2 + 4(\delta - 4)}}{\delta - 4}$$

and

$$\sigma_2 = \frac{-\pi - \sqrt{\pi^2 + 4(\delta - 4)}}{\delta - 4}.$$

(For the special case $\delta = 4$, $\sigma_1 = 2/\pi$.) However, σ_2 is invalid due to our constraint $0 < \sigma^2 < 1$. This can be seen as follows. First, $\sigma_2 < 0$ for $\delta > 4$ and hence invalid. When $2 < \delta < 4$, we have

$$\sigma_2 = \frac{\pi + \sqrt{\pi^2 - 4(4 - \delta)}}{4 - \delta} > \frac{\pi}{4 - \delta} > 1.$$

Hence, $F_2(0; \delta) = \sigma_1$. When $\alpha = 1$, (V.28) becomes:

$$\sigma^2 = \frac{4}{\delta}\left(2 + \sigma^2 - \int_0^{\frac{\pi}{2}} \frac{2\sin^2\theta + \sigma^2}{(\sin^2\theta + \sigma^2)^{\frac{1}{2}}}\mathrm{d}\theta\right), \quad \sigma^2 \in [0, 1].$$

It is straightforward to verify that $\sigma^2 = 0$ is a solution. Also, from Lemma 8 (ii), $\sigma^2 = 0$ is a also the unique solution. Hence, $F_2(1; \delta) = 0$. □

D. Proof of Lemma 3: $F_1^{-1}(\alpha) > F_2(\alpha; \delta)$ for $\delta > \delta_{\mathrm{AMP}}$

In Lemma 8, we have proved that $F_2(\alpha; \delta)$ is the unique globally attracting fixed point of ψ_2 in $\sigma^2 \in [0, 1]$ (for $\delta > 2$), and from (V.7) we have

$$\sigma^2 > F_2(\alpha; \delta) \Longleftrightarrow \psi_2(\alpha, \sigma^2; \delta) < \sigma^2, \quad \sigma^2 \in [0, 1]. \tag{V.29}$$

Here, our objective is to prove that $F_1^{-1}(\alpha) < F_2(\alpha; \delta)$ holds for any $\alpha \in (0, 1)$ when $\delta \geq \delta_{\mathrm{AMP}}$. From (V.29) and noting that $F_1^{-1}(\alpha) \leq \pi^2/16 < 1$ (from Lemma 9), our problem can be reformulated as proving the following inequality (for $\delta > \delta_{\mathrm{AMP}}$):

$$\psi_2(\alpha, F_1^{-1}(\alpha); \delta) < F_1^{-1}(\alpha), \quad \forall\alpha \in (0, 1). \tag{V.30}$$

Since $\psi_2(\alpha, F_1^{-1}(\alpha); \delta)$ is a strictly decreasing function of δ (see (II.2c)), it suffices to prove that (V.30) holds for $\delta = \delta_{\text{AMP}}$:

$$\psi_2(\alpha, F_1^{-1}(\alpha); \delta_{\text{AMP}}) < F_1^{-1}(\alpha), \quad \forall \alpha \in (0,1). \quad \text{(V.31)}$$

We now make some variable changes for (V.31). From (II.2a), ψ_1 in can be rewritten as the following for $\alpha > 0$:

$$\psi_1(\alpha, \sigma^2) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(\sin^2\theta + \frac{\sigma^2}{\alpha^2}\right)^{\frac{1}{2}}} \mathrm{d}\theta.$$

By definition, $F_1(\sigma^2)$ is the solution to $\alpha = \psi_1(\alpha, \sigma^2)$, and hence the following holds:

$$\alpha = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(\sin^2\theta + \frac{F_1^{-1}(\alpha)}{\alpha^2}\right)^{\frac{1}{2}}} \mathrm{d}\theta.$$

At this point, it is more convenient to make the following variable change:

$$s \triangleq \frac{\sqrt{F_1^{-1}(\alpha)}}{\alpha}, \quad \text{(V.32)}$$

from which we get

$$\alpha = \phi_1(s) \triangleq \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(\sin^2\theta + s^2)^{\frac{1}{2}}} \mathrm{d}\theta. \quad \text{(V.33)}$$

Notice that $\phi_1 : \mathbb{R}_+ \mapsto [0,1]$ is a monotonically decreasing function, and it defines a one-to-one map between α and s. From the above definitions, we have

$$F_1^{-1}(\alpha) = s^2\alpha^2 = s^2\phi_1^2(s), \quad \text{(V.34)}$$

where the first equality is from (V.32) and the second step from (V.33). Using the relationship in (V.34), we can reformulate the inequality in (V.31) into the following equivalent form:

$$\psi_2(\phi_1(s), s^2\phi_1^2(s); \delta_{\text{AMP}}) < s^2\phi_1^2(s), \quad \forall s > 0. \quad \text{(V.35)}$$

Substituting (V.33) and (II.2c) into (V.35) and after some manipulations, we can finally write our objective as:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(\sin^2\theta + s^2)^{\frac{1}{2}}} d\theta \cdot \int_0^{\frac{\pi}{2}} \frac{(1-\gamma s^2)\sin^2\theta + s^2}{(\sin^2\theta + s^2)^{\frac{1}{2}}} d\theta > 1, \quad \forall s > 0, \quad \text{(V.36)}$$

where we defined

$$\gamma \triangleq 1 - \frac{\delta_{\text{AMP}}}{4} = 2 - \frac{16}{\pi^2}. \quad \text{(V.37)}$$

In the next two subsections, we prove (V.36) for $s^2 > 0.07$ and $s^2 \le 0.07$ using different techniques.

(i) Case I: We make another variable change:

$$t \triangleq \frac{1}{s^2}.$$

Using the variable t, we can rewrite (V.36) into the following:

$$G(t) \triangleq \frac{g_1(t)}{g_2(t)} - \frac{1}{g_2^2(t)} \ge \gamma, \quad \forall t \in [0, 14.3), \quad \text{(V.38a)}$$

where γ is defined in (V.37), and

$$g_1(t) \triangleq \int_0^{\frac{\pi}{2}} (1 + t\sin^2\theta)^{\frac{1}{2}} d\theta, \quad \text{(V.38b)}$$

$$g_2(t) \triangleq \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1 + t\sin^2\theta)^{\frac{1}{2}}} d\theta. \quad \text{(V.38c)}$$

Notice that if we could prove (V.38a) for $t < 14.3$, we would have proved (V.36) for $s^2 > 0.07$, since $14.3 > 1/0.07 \approx 14.28$. For the ease of later discussions, we define

$$g_3(t) \triangleq \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta}{(1 + t\sin^2\theta)^{\frac{3}{2}}} d\theta,$$

$$g_4(t) \triangleq \int_0^{\frac{\pi}{2}} \frac{\sin^6\theta}{(1 + t\sin^2\theta)^{\frac{5}{2}}} d\theta.$$

The following identities related to $\{g_1(t), g_2(t), g_3(t), g_4(t)\}$ will be used in our proof:

$$\begin{aligned} g_1'(t) &= \frac{1}{2} g_2(t), \\ g_2'(t) &= -\frac{1}{2} g_3(t), \\ g_3'(t) &= -\frac{3}{2} g_4(t). \end{aligned} \quad \text{(V.39)}$$

We now prove (V.38a). First, it is straightforward to verify that equality holds for (V.38a) at $t = 0$, i.e.,

$$G(0) = \gamma. \quad \text{(V.40)}$$

Hence, to prove that $G(t) \ge \gamma$ for $t \in [0, 14.3)$, it is sufficient to prove that $G(t)$ is an increasing function of t on $t \in [0, 14.3)$. To this end, we calculate the derivative of $G(t)$:

$$\begin{aligned} G'(t) &= \frac{g_1'(t)g_2(t) - g_1(t)g_2'(t)}{g_2^2(t)} - \left(\frac{-2\, g_2'(t)}{g_2^3(t)}\right) \\ &\overset{(a)}{=} \frac{\frac{1}{2}g_2^2(t) + \frac{1}{2}g_1(t)g_3(t)}{g_2^2(t)} - \frac{g_3(t)}{g_2^3(t)} \\ &= 1 + \frac{1}{2}\frac{g_1(t)g_3(t)}{g_2^2(t)} - \frac{g_3(t)}{g_2^3(t)} \\ &= \frac{1}{2}\frac{g_3(t)}{g_2^3(t)}\Bigg(\underbrace{\frac{g_2^3(t)}{g_3(t)}}_{G_1(t)} + \underbrace{g_1(t)g_2(t)}_{G_2(t)} - 2\Bigg), \end{aligned}$$

where step (a) follows from the identities listed in (V.39). Since $g_3(t) > 0$, we have

$$G'(t) > 0 \Longleftrightarrow G_1(t) + G_2(t) - 2 > 0.$$

It remains to prove that $G_1(t) + G_2(t) - 2 > 0$ for $t < 14.3$. Our numerical results suggest that $G_1(t) + G_2(t)$ is a monotonically decreasing function for $t > 0$, and $G_1(t) + G_2(t) \to 2$ as $t \to \infty$. However, directly proving the monotonicity of $G_1(t) + G_2(t)$ seems to be quite complicated. We use a different strategy here. We will prove that (at the end of this section)

- $G_1(t)$ is monotonically increasing;
- $G_2(t)$ is monotonically decreasing.

As a consequence, the following hold true for any $c_2 > c_1 > 0$:

$$G_1(t)+G_2(t)-2 \geq G_1(c_1)+G_2(c_2)-2, \quad \forall t \in [c_1, c_2].$$

Hence, if we verify that $G_1(c_1)+G_2(c_2)-2 > 0$, we will be proving the following:

$$G_1(t) + G_2(t) - 2 > 0, \quad \forall t \in [c_1, c_2].$$

To this end, we verify that $G_1(c_1) + G_2(c_2) - 2 > 0$ hold for a sequence of c_1 and c_2: $[c_1, c_2] = [0, 0.49]$, $[c_1, c_2] = [0.49, 1.08]$, $[c_1, c_2] = [1.08, 1.78]$, $[c_1, c_2] = [1.78, 2.56]$, $[c_1, c_2] = [2.56, 3.47]$, $[c_1, c_2] = [3.47, 4.47]$, $[c_1, c_2] = [4.47, 5.56]$, $[c_1, c_2] = [5.56, 6.77]$, $[c_1, c_2] = [6.67, 8.08]$, $[c_1, c_2] = [8.08, 9.5]$, $[c_1, c_2] = [9.5, 11]$, $[c_1, c_2] = [11, 12.6]$, $[c_1, c_2] = [12.6, 14.3]$. Combining all the above results proves

$$G_1(t) + G_2(t) - 2 > 0, \quad \forall t \in [0, 14.3].$$

From the above discussions, it only remains to prove the monotonicity of $G_1(t)$ and $G_2(t)$. Consider $G_1(t)$ first:

$$\begin{aligned} G_1'(t) &= \left(\frac{g_2^3(t)}{g_3(t)}\right)' \\ &= \frac{3g_2^2(t)g_2'(t)g_3(t) - g_2^3(t)g_3'(t)}{g_3^2(t)} \\ &= \frac{-\frac{3}{2}g_2^2(t)g_3^2(t) + \frac{3}{2}g_2^3(t)g_4(t)}{g_3^2(t)} \\ &= -\frac{3}{2}g_2^2(t) + \frac{3}{2}\frac{g_2^3(t)g_4(t)}{g_3^2(t)} \\ &= \frac{3}{2}\frac{g_2^2(t)}{g_3^2(t)} \cdot [-g_3^2(t) + g_2(t)g_4(t)]. \end{aligned} \tag{V.41}$$

Applying the Cauchy-Schwarz inequality yields:

$$\begin{aligned} &g_2(t)g_4(t) \\ &\quad = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+t\sin^2\theta)^{\frac{1}{2}}}d\theta \int_0^{\frac{\pi}{2}} \frac{\sin^6\theta}{(1+t\sin^2\theta)^{\frac{5}{2}}}d\theta \\ &\quad \geq \left(\int_0^{\frac{\pi}{2}} \frac{\sin^4\theta}{(1+t\sin^2\theta)^{\frac{3}{2}}}d\theta\right)^2 \\ &\quad = g_3^2(t). \end{aligned} \tag{V.42}$$

Combining (V.41) and (V.42), we proved that $G_1'(t) \geq 0$, and therefore $G_1(t)$ is monotonically increasing. For $G_2(t)$, we have

$$\begin{aligned} G_2'(t) &= g_1'(t)g_2(t) + g_1(t)g_2'(t) \\ &= \frac{1}{2}g_2^2(t) + g_1(t)\left(-\frac{1}{2}g_3(t)\right) \\ &= \frac{1}{2}[g_2^2(t) - g_1(t)g_3(t)]. \end{aligned}$$

Again, using Cauchy-Schwarz we have

$$\begin{aligned} &g_1(t)g_3(t) \\ &\quad = \int_0^{\frac{\pi}{2}} (1+t\sin^2\theta)^{\frac{1}{2}}d\theta \cdot \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta}{(1+t\sin^2\theta)^{\frac{3}{2}}}d\theta \\ &\quad \geq \left(\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+t\sin^2\theta)^{\frac{1}{2}}}d\theta\right)^2 \\ &\quad = g_2^2(t). \end{aligned}$$

Combining the previous two equations leads to $G_2'(t) \geq 0$, which completes our proof.

(ii) Case II: We next prove (V.36) for $s^2 \leq 0.07$, which is based on a different strategy. Some manipulations of the RHS of (V.36) yields:

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(\sin^2\theta+s^2)^{\frac{1}{2}}}d\theta \cdot \int_0^{\frac{\pi}{2}} \frac{\left(1-\gamma s^2\right)\sin^2\theta + s^2}{(\sin^2\theta+s^2)^{\frac{1}{2}}}d\theta \\ &\qquad = \frac{E(x)T(x)}{x} - \frac{\gamma(1-x)T^2(x)}{x^2}, \end{aligned} \tag{V.43a}$$

where $E(\cdot)$, $K(\cdot)$ and $T(\cdot)$ are elliptic integrals defined in (A.1), γ is a constant defined in (V.37), and x is a new variable:

$$x \triangleq \frac{1}{1+s^2}. \tag{V.43b}$$

From our reformulation in (V.43), the inequality in (V.36) for $s^2 < 0.07$ becomes

$$\frac{E(x)T(x)}{x} - \gamma\frac{(1-x)T^2(x)}{x^2} > 1, \quad x \in [0.93, 1). \tag{V.44}$$

Note that $0.93 < 1/(1+0.07)$ and thus proving the above inequality for $x \in [0.93, 1)$ is sufficient to prove the original inequality for $s^2 \leq 0.07$ (note that $x \triangleq 1/(1+s^2)$, see (V.43b)).

With some further calculations, (V.44) can be reformulated as

$$\frac{x}{T^2(x)}\frac{E(x)T(x)-x}{(1-x)} > \gamma, \quad x \in [0.93, 1). \tag{V.45}$$

The following inequality is due to [59, eq. (1)]

$$T(x) < x < 1, \quad \forall x \in (0, 1).$$

Hence,

$$\frac{x}{T^2(x)}\frac{E(x)T(x)-x}{(1-x)} > \frac{E(x)T(x)-x}{1-x}, \quad \forall x \in (0, 1),$$

and to prove (V.45) it suffices to prove the following

$$\frac{E(x)T(x)-x}{1-x} > \gamma, \quad \forall x \in [0.93, 1). \tag{V.46}$$

To this end, we will prove that the LHS of (V.46) is a strictly increasing function of $x \in [0.93, 1)$. If this is true, we would have

$$\begin{aligned} &\frac{E(x)T(x)-x}{1-x} \\ &\quad > \frac{E(x)T(x)-x}{1-x}\Big|_{x=0.93} \approx 0.385 \\ &\quad > \gamma = 2 - \frac{16}{\pi^2} \approx 0.3789, \quad x \in [0.93, 1). \end{aligned}$$

We next prove the monotonicity of $\frac{E(x)T(x)-x}{1-x}$. From the identities in Lemma 18, we derive the following

$$[E(x)T(x)-x]' = \frac{E^2(x)-2(1-x)E(x)K(x)+(1-x)K^2(x)}{2\,x}-1.$$

Hence, to prove that $\frac{E(x)T(x)-x}{1-x}$ is monotonically increasing, it is sufficient to prove the following inequality

$$\left(\frac{E^2(x)-2(1-x)E(x)K(x)+(1-x)K^2(x)}{2x}-1\right)(1-x)-[E(x)T(x)-x](-1)>0. \quad \text{(V.47)}$$

Now, substituting $T(x)=E(x)-(1-x)K(x)$ into (V.47) and after some manipulations, we finally reformulate the inequality to be proved into the following form:

$$T(x)^2 > 2\,x - xE^2(x).$$

It can be verified that equality holds at $x=1$. We next prove that $T(x)^2+xE(x)^2-2\,x$ is monotonically decreasing on $[0.93,1)$. We differentiate once more:

$$(T(x)^2+xE(x)^2-2\,x)' = 2\,E(x)^2-(1-x)K(x)^2-2.$$

Our problem boils down to proving $2\,E(x)^2-(1-x)K(x)^2-2<0$ for $x\in[0.93,1)$. We can verify that $2\,E(x)^2-(1-x)K(x)^2-2=0$ holds at $x=1$. We finish by showing that $2\,E(x)^2-(1-x)K(x)^2-2$ is monotonically increasing in $x\in[0.93,1)$. To this end, we differentiate again:

$$\begin{aligned}[2E(x)^2-(1-x)K(x)^2-2]' &= \frac{K(x)^2-3E(x)K(x)+2E(x)^2}{x}\\ &= \frac{\left[K(x)-\frac{3}{2}E(x)\right]^2-\frac{1}{2}E(x)^2}{x}. \end{aligned} \quad \text{(V.48)}$$

We note that $K(x)-\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)E(x)$ is a monotonically increasing function in (0,1) since $K(x)$ is monotonically increasing and $E(x)$ is monotonically decreasing. We verify that $K(x)-\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)E(x)>0$ when $x\geq 0.93$. Hence,

$$K(x)-\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)E(x)>0, \quad \forall x\in[0.93,1),$$

and therefore

$$\left(K(x)-\frac{3}{2}E(x)\right)^2 > \frac{1}{2}E(x)^2, \quad \forall x\in[0.93,1). \quad \text{(V.49)}$$

Substituting (V.49) into (V.48), we prove that $[2\,E(x)^2-(1-x)K(x)^2-2]'>0$ for $x\in[0.93,1)$, which completes the proof.

E. Proof of Lemma 5: Behavior of the SE in $\mathcal{R}_1\cup\mathcal{R}_2$

First, we introduce a function that will be crucial for our proof.

Definition 5: Define

$$L(\alpha;\delta)\triangleq\frac{4}{\delta}\left(1-\frac{\phi_2^2(\phi_1^{-1}(\alpha))}{4\left[1+(\phi_1^{-1}(\alpha))^2\right]}\right), \quad \alpha\in(0,1), \quad \text{(V.50)}$$

where $\phi_1:\mathbb{R}_+\mapsto[0,1]$ and $\phi_2:\mathbb{R}_+\mapsto\mathbb{R}_+$ below:

$$\phi_1(s)\triangleq\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\left(\sin^2\theta+s^2\right)^{\frac{1}{2}}}\mathrm{d}\theta, \quad \text{(V.51a)}$$

$$\phi_2(s)\triangleq\int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta+s^2}{\left(\sin^2\theta+s^2\right)^{\frac{1}{2}}}\mathrm{d}\theta, \quad \text{(V.51b)}$$

where ϕ_1^{-1} is the inverse functions of ϕ_1. The existence of ϕ_1^{-1} follows from its monotonicity, which can be seen from its definition.

In the following, we list some preliminary properties of $L(\alpha;\delta)$. The main proof for Lemma 5 comes afterwards.

- **Preliminaries:**
  The following lemma helps us clarify the importance of L in the analysis of the dynamics of SE:
  Lemma 10: For any $\alpha>0$, $\sigma^2>0$ and $\delta>0$, the following holds:

$$L\left[\psi_1(\alpha,\sigma^2);\delta\right]\leq\psi_2(\alpha,\sigma^2;\delta), \quad \text{(V.52)}$$

where ψ_1 and ψ_2 are the SE maps defined in (II.2)*, and $L(\alpha;\delta)$ is defined in* (V.50)*.*

Proof: Define $\mathcal{X}\triangleq\{(\alpha,\sigma^2)|\alpha>0,\sigma^2>0\}$. Let $\mathcal{Y}$ be the image of $\mathcal{X}$ under the SE map in (II.2). We will prove that the following holds for an arbitrary $C\in[0,1]$:

$$L\,(C;\delta)=\min_{(\hat{\alpha},\hat{\sigma}^2)\in\mathcal{X}}\ \psi_2(\hat{\alpha},\hat{\sigma}^2;\delta), \quad \text{(V.53)}$$

where $(\hat{\alpha},\hat{\sigma}^2)$ satisfies the constraint

$$\psi_1(\hat{\alpha},\hat{\sigma}^2)=C.$$

If (V.53) holds, we would have proved (V.52). To see this, consider arbitrary (α,σ^2) such that $\psi_1(\alpha,\sigma^2)=C$. Then, we have

$$L\left[\psi_1(\alpha,\sigma^2);\delta\right]\overset{(a)}{=}\min_{(\hat{\alpha},\hat{\sigma}^2)}\ \psi_2(\hat{\alpha},\hat{\sigma}^2;\delta)\overset{(b)}{\leq}\psi_2(\alpha,\sigma^2;\delta),$$

where step (a) follows from (V.53) and $\psi_1(\alpha,\sigma^2)=C$, and step (b) holds since the choice $\hat{\alpha}=\alpha$ and $\hat{\sigma}^2=\sigma^2$ is feasible for the constraint $\psi_1(\hat{\alpha},\hat{\sigma}^2)=\psi_1(\alpha,\sigma^2)$. This is precisely (V.52).

We now prove (V.53). From (II.2a) we have

$$\psi_1(\alpha,\sigma^2)=\int_0^{\pi/2}\frac{\alpha\sin^2\theta}{(\alpha^2\sin^2\theta+\sigma^2)^{1/2}}d\theta.$$

Furthermore, from the definition of ϕ_1 in (V.51a) we have

$$\psi_1(\hat{\alpha},\hat{\sigma}^2)=\phi_1\left(\frac{\hat{\sigma}}{\hat{\alpha}}\right)=C\Longrightarrow s\triangleq\frac{\hat{\sigma}}{\hat{\alpha}}=\phi_1^{-1}(C). \quad \text{(V.54)}$$

Similarly, from (II.2c), i.e. the definition of ψ_2, and the definition of ϕ_2 in (V.51b), we can express $\psi_2(\hat{\alpha}, \hat{\sigma}^2; \delta)$ as

$$\begin{aligned}\psi_2(\hat{\alpha}, \hat{\sigma}^2; \delta) &= \frac{4}{\delta}\left[\hat{\alpha}^2 + \hat{\sigma}^2 + 1 - \hat{\alpha}\cdot\phi_2\left(\frac{\hat{\sigma}}{\hat{\alpha}}\right)\right] \\ &= \frac{4}{\delta}\left[(1+s^2)\hat{\alpha}^2 + 1 - \hat{\alpha}\cdot\phi_2(s)\right].\end{aligned}$$

From (V.54), we see that fixing $\psi_1(\hat{\alpha}, \hat{\sigma}^2) = C$ is equivalent to fixing $s = \phi_1^{-1}(C)$. Further, for a fixed s, $\psi_2(\hat{\alpha}, \hat{\sigma}^2)$ is a quadratic function of $\hat{\alpha}$, and the minimum is achieved at

$$\hat{\alpha}_{\min} = \frac{\phi_2(s)}{2(1+s^2)} = \frac{\phi_2(\phi_1^{-1}(C))}{2\left[1 + \left(\phi_1^{-1}(C)\right)^2\right]},$$

and $\psi_2(\hat{\alpha}_{\min}, \hat{\sigma}^2; \delta)$ is

$$\begin{aligned}\psi_2(\hat{\alpha}_{\min}, \hat{\sigma}^2; \delta) &= \frac{4}{\delta}\left(1 - \frac{\phi_2^2(s)}{4(1+s^2)}\right) \\ &= \frac{4}{\delta}\left(1 - \frac{\phi_2^2(\phi_1^{-1}(C))}{4\left(1 + \left[\phi_1^{-1}(C)\right]^2\right)}\right) \\ &= L\left(C; \delta\right),\end{aligned}$$

where the last step is from the definition of L is (V.50). This completes the proof. □

To understand the implication of this lemma, let us consider the t^{th} iteration of the SE:

$$\begin{aligned}\alpha_{t+1} &= \psi_1(\alpha_t, \sigma_t^2), \\ \sigma_{t+1}^2 &= \psi_2(\alpha_t, \sigma_t^2; \delta).\end{aligned}$$

Note that according to Lemma 10, no matter where (α_t, σ_t^2) is, $(\alpha_{t+1}, \sigma_{t+1}^2)$ will fall above the curve defined by $\sigma^2 = L(\alpha; \delta)$. This function is a key component in the dynamics of AMP.A. Before we proceed further, we discuss two main properties of the function $L(\alpha; \delta)$.

Lemma 11: $L(\alpha; \delta)$ is a strictly decreasing function of $\alpha \in (0, 1)$.

Proof: Recall from (V.50) that $L(\alpha; \delta)$ is defined as

$$\begin{aligned}L(\alpha; \delta) &\triangleq \frac{4}{\delta}\left(1 - \frac{\phi_2^2(\phi_1^{-1}(\alpha))}{4(1 + (\phi_1^{-1}(\alpha))^2)}\right) \\ &= \frac{4}{\delta}\left(1 - I_2[\phi_1^{-1}(\alpha)]\right),\end{aligned}$$

where $I_2 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined as

$$I_2(s) \triangleq \frac{\phi_2^2(s)}{4(1+s^2)}. \tag{V.55}$$

From (V.51a), it is easy to see that $\phi_1(s)$ is a decreasing function. Hence, to prove that $L(\alpha; \delta)$ is a decreasing function of α, it suffices to prove that $I_2(s)$ is strictly decreasing.

Substituting (V.51b) into (V.55) yields:

$$\begin{aligned}I_2(s) &= \frac{\phi_2^2(s)}{4(1+s^2)} \\ &= \frac{1}{4(1+s^2)}\left(\int_0^{\frac{\pi}{2}} \frac{2\sin^2\theta + s^2}{(\sin^2\theta + s^2)^{\frac{1}{2}}}\right)^2 \\ &\overset{(a)}{=} \frac{1}{4}\left[2E\left(\frac{1}{1+s^2}\right) - \frac{s^2}{1+s^2}K\left(\frac{1}{1+s^2}\right)\right]^2 \\ &= \frac{1}{4}\left[2E(x) - (1-x)K(x)\right]^2,\end{aligned}$$

where step (a) is obtained through similar calculations as those in (A.5), and in the last step we defined $x = \frac{1}{1+s^2}$. Hence, to prove that $I_2(s)$ is a decreasing function of s, it suffices to prove that $[2E(x) - (1-x)K(x)]^2$ is an increasing function of x. Further, $2E(x) - (1-x)K(x) = T(x) + E(x) > 0$ (form the definition of $T(x)$ in (A.1)), our problem reduces to proving that $2E(x) - (1-x)K(x)$ is increasing. To this end, differentiation yields

$$\begin{aligned}[2E(x) - (1-x)K(x)]' &\overset{(a)}{=} \frac{E(x) - (1-x)K(x)}{2x} \\ &\overset{(b)}{=} \frac{1}{2}T(x) \overset{(c)}{>} 0,\end{aligned}$$

where (a) is from the differentiation identities in Lemma 18, (b) is from (A.1), and $T(x) > 0$ follows from Lemma 18 (ii) together with the fact that $T(0) = 0$. □

The next lemma compares the function $L(\alpha; \delta)$ with $F_1^{-1}(\alpha)$.

Lemma 12: If $\delta > \delta_{\text{AMP}}$, then

$$F_1^{-1}(\alpha) > L(\alpha; \delta), \quad \forall \alpha \in (0, 1).$$

Proof: We prove by contradiction. Suppose that $L(\hat{\alpha}; \delta) \geq F_1^{-1}(\hat{\alpha})$ at some $\hat{\alpha} \in (0, 1)$. If this is the case, then there exists a $\hat{\sigma}^2$ such that

$$F_1^{-1}(\hat{\alpha}) \leq \hat{\sigma}^2 \leq L(\hat{\alpha}; \delta). \tag{V.56}$$

Since F_1 is a decreasing function (see Lemma 9), the first inequality implies that $\hat{\alpha} \geq F_1(\hat{\sigma}^2)$. Then, based on the global attractiveness property in Lemma 7 (iii), we have

$$\psi_1(\hat{\alpha}, \hat{\sigma}^2) \leq \hat{\alpha}. \tag{V.57}$$

Further, Lemma 3 shows that $F_1^{-1}(\hat{\alpha}) > F_2(\hat{\alpha}; \delta)$ for $\delta > \delta_{\text{AMP}}$, and using (V.56) we also have $\hat{\sigma}^2 \geq F_1^{-1}(\hat{\alpha}) > F_2(\hat{\alpha}; \delta)$. Also, from (V.56),

$$\hat{\sigma}^2 \leq L(\hat{\alpha}; \delta) \overset{(a)}{<} L(0; \delta) = \frac{4}{\delta}\left(1 - \frac{\pi^2}{16}\right) < \frac{4}{\delta} \leq \sigma_{\max}^2,$$

where (a) is due to the monotonicity of $L(\alpha; \delta)$ (see Lemma 11). From the above discussions, $F_2(\hat{\alpha}; \delta) < \hat{\sigma}^2 < \sigma_{\max}^2$. We then have (for $\delta > \delta_{\text{AMP}}$):

$$\psi_2(\hat{\alpha}, \hat{\sigma}^2; \delta) \overset{(a)}{<} \hat{\sigma}^2 \overset{(b)}{\leq} L(\hat{\alpha}; \delta) \overset{(c)}{\leq} L\left[\psi_1(\hat{\alpha}, \hat{\sigma}^2); \delta\right], \tag{V.58}$$

where step (a) follows from the global attractiveness property in Lemma 8 (iv), step (b) is due to the hypothesis in (V.56), step (c) is from (V.57) together with the monotonicity of $L(\alpha;\delta)$ (see Lemma 11). Note that (V.58) shows that $\psi_2(\hat{\alpha},\hat{\sigma}^2;\delta) < L\left[\psi_1(\hat{\alpha},\hat{\sigma}^2);\delta\right]$, which contradicts Lemma 10, where we proved that $\psi_2(\alpha,\sigma^2;\delta) \geq L\left[\psi_1(\alpha,\sigma^2);\delta\right]$ for any $\alpha > 0$, $\sigma^2 > 0$ and $\delta > 0$. Hence, we must have that $L(\alpha;\delta) < F_1^{-1}(\alpha)$ for any $\alpha \in (0,1)$. □

Lemma 13: The following holds for any $\alpha \in (0,1)$ *and* $\delta > 0$*,*

$$L(\alpha;\delta) > \frac{4}{\delta}\left(1-\frac{\pi^2}{16}-\frac{1}{2}\alpha^2\right), \tag{V.59}$$

where $L(\alpha,\delta)$ *is defined in* (V.50)*.*

Proof: From (V.50), proving (V.59) is equivalent to proving:

$$1-\frac{\phi_2^2(\phi_1^{-1}(\alpha))}{4\left[1+(\phi_1^{-1}(\alpha))^2\right]} > 1-\frac{\pi^2}{16}-\frac{1}{2}\alpha^2, \quad \forall \alpha \in (0,1), \tag{V.60}$$

where $\phi_1 : [0,\infty) \mapsto [0,1]$ and $\phi_2 : [0,\infty) \mapsto [0,\infty)$ are defined as (see (V.51a) and (V.51b)):

$$\phi_1(s) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(\sin^2\theta+s^2\right)^{\frac{1}{2}}}\mathrm{d}\theta, \tag{V.61a}$$

$$\phi_2(s) = \int_0^{\frac{\pi}{2}} \frac{2\sin^2\theta+s^2}{\left(\sin^2\theta+s^2\right)^{\frac{1}{2}}}\mathrm{d}\theta. \tag{V.61b}$$

We make a variable change:

$$\alpha = \phi_1(s).$$

Simple calculations show that (V.60) can be reformulated as the following

$$\frac{1}{1+s^2}\phi_2^2(s) < \frac{\pi^2}{4}+2\phi_1^2(s), \quad s \in (0,\infty). \tag{V.62}$$

Let us further define

$$\phi_3(s) \equiv \int_0^{\frac{\pi}{2}} (\sin^2\theta+s^2)^{\frac{1}{2}}\mathrm{d}\theta. \tag{V.63}$$

From (V.61) and (V.63), we have

$$\phi_2(s) = \phi_1(s)+\phi_3(s),$$

and (V.62) can be reformulated as

$$[\phi_1(s)+\phi_3(s)]^2-(1+s^2)\left[\frac{\pi^2}{4}+2\phi_1^2(s)\right] < 0. \tag{V.64}$$

To this end, we can write the LHS of (V.64) into a quadratic form of $\phi_1(s)$:

$$\begin{aligned}
&[\phi_1(s)+\phi_3(s)]^2-(1+s^2)\left[\frac{\pi^2}{4}+2\phi_1^2(s)\right]\\
&= \phi_1^2(s)+\phi_3^2(s)+2\phi_1(s)\phi_3(s)-(1+s^2)\\
&\quad\times\left[\frac{\pi^2}{4}+2\phi_1^2(s)\right]\\
&= -(1+2s^2)\phi_1^2(s)+2\phi_1(s)\phi_3(s)-\frac{\pi^2}{4}(1+s^2)+\phi_3^2(s).
\end{aligned}$$

Hence, to prove that this quadratic form is negative everywhere, it suffices to prove that the discriminant is negative, i.e.,

$$4\phi_3^2(s)+4(1+2s^2)\left[-\frac{\pi^2}{4}(1+s^2)+\phi_3^2(s)\right] < 0,$$

or

$$\phi_3^2(s) < \frac{\pi^2}{8}(1+2\,s^2).$$

Finally, by Cauchy-Schwarz we have

$$\begin{aligned}
\phi_3^2(s) &= \left[\int_0^{\frac{\pi}{2}} (\sin^2\theta+s^2)^{\frac{1}{2}}\mathrm{d}\theta\right]^2\\
&\leq \int_0^{\frac{\pi}{2}} 1\mathrm{d}\theta \cdot \int_0^{\frac{\pi}{2}} \left(\sqrt{\sin^2\theta+s^2}\right)^2\mathrm{d}\theta\\
&= \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{\pi}{2}s^2\right) = \frac{\pi^2}{8}(1+2\,s^2),
\end{aligned}$$

which completes our proof. □

Lemma 14: For any $\alpha \in [0,1]$*,* $\psi_2(\alpha,\sigma^2;\delta_{\mathrm{AMP}})$ *is an increasing function of* σ^2 *on* $\sigma^2 \in [L(\alpha;\delta_{\mathrm{AMP}}),\infty)$*, where the function* $L(\alpha;\delta)$ *is defined in* (5)*.*

Proof: From Lemma 8 (v), the case $\alpha > \alpha_* \approx 0.53$ is trivial since then $\psi_2(\sigma^2,\alpha;\delta_{\mathrm{AMP}})$ is strictly increasing in $\sigma^2 \in \mathbb{R}_+$. In the rest of this proof, we assume that $\alpha < \alpha_*$. We have derived in (V.10) that

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} > 0 \Longleftrightarrow \alpha > \frac{1}{2\sqrt{1+s^2}}E\left(\frac{1}{1+s^2}\right) = f(s), \tag{V.65}$$

where

$$s \triangleq \frac{\sigma}{\alpha}.$$

Hence, the result of Lemma 14 can be reformulated as proving the following:

$$\alpha > f(s), \quad \forall s \geq \frac{\sqrt{L(\alpha;\delta_{\mathrm{AMP}})}}{\alpha}, \ \alpha \in [0,\alpha_*).$$

We proceed in three steps:

(i) In Lemma 13, we proved that the following holds for any $\alpha \in [0,1]$:

$$\begin{aligned}
L(\alpha;\delta_{\mathrm{AMP}}) &\geq \hat{L}(\alpha,\delta_{\mathrm{AMP}})\\
&\triangleq \frac{4}{\delta_{\mathrm{AMP}}}\left(1-\frac{\pi^2}{16}-\frac{1}{2}\alpha^2\right).
\end{aligned} \tag{V.66}$$

For convenience, define

$$\hat{s}(\alpha) \triangleq \frac{\sqrt{\hat{L}(\alpha;\delta_{\mathrm{AMP}})}}{\alpha}. \tag{V.67}$$

(ii) We prove that $f(s)$ is monotonically decreasing on $s \in \left[\hat{s}(\alpha),\infty\right)$ for $\alpha < \alpha_*$.

(iii) We prove that the following holds for $\alpha < \alpha_*$:

$$\alpha > f(\hat{s}(\alpha)).$$

Clearly, (V.65) follows from the above claims. Here, we introduce the function $\hat{L}$ since $\hat{L}$ has a simple closed-form formula and is easier to manipulate than $L(\alpha)$.

We next prove step (ii). From (V.19), it suffices to prove that

$$\hat{s}(\alpha) > s_*, \quad \forall \alpha < \alpha_*,$$

where s_* and α_* are defined in (V.24) and (V.23) respectively. To this end, we note that the following holds for $\alpha < \alpha_*$:

$$\hat{s}(\alpha) = \frac{\sqrt{\hat{L}(\alpha; \delta_{\text{AMP}})}}{\alpha} > \frac{\sqrt{\hat{L}(\alpha_*; \delta_{\text{AMP}})}}{\alpha_*} \approx 1.18,$$

where the inequality follows from the fact that $\hat{L}$ in (V.66) is strictly decreasing in α, and the last step is calculated from (V.66) and $\alpha_* \approx 0.527$. Finally, numerical evaluation of (V.24) shows that $s_* \approx 0.458$. Hence, $\hat{s}(\alpha) > s_*$, which completes the proof.

We next prove step (iii). First, simple manipulations yields

$$\hat{s}^2(\alpha) \overset{(a)}{=} \frac{\hat{L}(\alpha)}{\alpha^2} \overset{(b)}{=} \frac{4}{\delta_{\text{AMP}}}\left[\left(1 - \frac{\pi^2}{16}\right)\cdot\frac{1}{\alpha^2} - \frac{1}{2}\right], \tag{V.68}$$

where (a) is from the definition of $\hat{s}(\alpha)$ in (V.67) and (b) is due to (V.66). Using (V.68), we further obtain

$$\alpha = \sqrt{\frac{16-\pi^2}{4\delta_{\text{AMP}}\hat{s}^2(\alpha)+8}}. \tag{V.69}$$

Now, from (V.69) and (V.17b), we have

$$\alpha - f(\hat{s}(\alpha)) > 0 \Longleftrightarrow \sqrt{\frac{16-\pi^2}{4\delta_{\text{AMP}}\hat{s}^2(\alpha)+8}} - \frac{1}{2\sqrt{1+\hat{s}^2(\alpha)}}E\left(\frac{1}{1+\hat{s}^2(\alpha)}\right) > 0. \tag{V.70}$$

We prove (V.70) by showing that the following stronger result holds:

$$\sqrt{\frac{16-\pi^2}{4\delta_{\text{AMP}}t^2+8}} - \frac{1}{2\sqrt{1+t^2}}E\left(\frac{1}{1+t^2}\right) > 0, \quad \forall t \in \mathbb{R}_+. \tag{V.71}$$

For convenience, we make a variable change:

$$x \overset{\Delta}{=} \frac{1}{1+t^2}.$$

With some straightforward calculations, we can rewrite (V.71) as

$$E(x) < \sqrt{\frac{16-\pi^2}{\delta_{\text{AMP}}(1-x)+2\,x}}$$

The following upper bound on $E(x)$ is due to [60, eq. (1.2)]:

$$E(x) < \frac{\pi}{2}\sqrt{1-\frac{x}{2}}, \quad \forall x \in (0,1].$$

Hence, it is sufficient to prove that

$$\frac{\pi}{2}\sqrt{1-\frac{x}{2}} < \sqrt{\frac{16-\pi^2}{\delta_{\text{AMP}}(1-x)+2\,x}},$$

which can be reformulated as

$$\left(1-\frac{x}{2}\right)(\delta_{\text{AMP}} - (\delta_{\text{AMP}}-2)x) < \frac{4}{\pi^2}(16-\pi^2) = \delta_{\text{AMP}},$$

where the second equality follows from the definition $\delta_{\text{AMP}} = \frac{64}{\pi^2} - 4$. The above inequality holds since $0 < 1 - \frac{x}{2} < 1$ and $0 < \delta_{\text{AMP}} - (\delta_{\text{AMP}} - 2)x < \delta_{\text{AMP}}$. This completes the proof. □

Lemma 15: For any $\alpha \in [0,1]$*,* $\psi_2(\alpha, L(\alpha;\delta);\delta)$ *is a strictly decreasing function of* $\delta > 0$*, where* $L(\alpha;\delta)$ *is defined in* (V.50)*.*

Proof: From the definition of $L(\alpha;\delta)$ in (V.50), we can write

$$\psi_2(\alpha, L(\alpha;\delta);\delta) = \psi_2\left(\alpha, \frac{1}{\delta}\bar{\sigma}^2;\delta\right),$$

where (note that $\bar{\sigma}$ is not the conjugate of σ)

$$\bar{\sigma}^2 \overset{\Delta}{=} 4\left(1 - \frac{\phi_2^2(\phi_1^{-1}(\alpha))}{4\left[1+(\phi_1^{-1}(\alpha))^2\right]}\right).$$

A key observation here is that $\bar{\sigma}^2$ does not depend on δ. Clearly, Lemma 15 is implied by the following stronger result:

$$\frac{\partial\psi_2\left(\alpha, \frac{1}{\delta}\bar{\sigma}^2;\delta\right)}{\partial\delta} < 0, \quad \forall\bar{\sigma}^2 > 0,\ \alpha > 0,\ \delta > 0,$$

which we will prove in the sequel. For convenience, we define

$$\bar{s} \overset{\Delta}{=} \frac{\bar{\sigma}}{\alpha}, \quad \gamma \overset{\Delta}{=} \frac{1}{\delta} \text{ and } s = \sqrt{\gamma}\,\bar{s}. \tag{V.72}$$

Using these new variables, we have

$$\begin{aligned}
&\psi_2\left(\alpha, \frac{1}{\delta}\bar{\sigma}^2;\delta\right)\\
&= \psi_2\left(\alpha, \gamma\,\bar{\sigma}^2;\gamma^{-1}\right)\\
&= 4\gamma\left((1+\gamma\,\bar{s}^2)\alpha^2 + 1 - \alpha\int_0^{\frac{\pi}{2}}\frac{2\sin^2\theta+\gamma\,\bar{s}^2}{\left(\sin^2\theta+\gamma\,\bar{s}^2\right)^{\frac{1}{2}}}\mathrm{d}\theta\right),
\end{aligned}$$

where the last equality is from the definition of ψ_2 in (II.2c). It remains to prove that $\psi_2\left(\alpha, \gamma\,\bar{\sigma}^2;\gamma^{-1}\right)$ is an increasing function of γ. The partial derivative of $\psi_2(\alpha, \sigma^2;\delta)$ w.r.t. γ is given by (V.73) (as shown on the top of the next page), where in step (a) we used the relationship $s^2 = \gamma\,\bar{s}^2$ (see (V.72)), and step (b) is from the identities in (A.5). From (V.73), we see that $\frac{\partial\psi_2\left(\alpha,\gamma\,\bar{\sigma}^2;\gamma^{-1}\right)}{\partial\gamma}$ is a quadratic function of α. Therefore, to prove $\frac{\partial\psi_2\left(\alpha,\gamma\,\bar{\sigma}^2;\gamma^{-1}\right)}{\partial\gamma} > 0$, it suffices to show that the discriminant is negative:

$$\left(\frac{(5s^2+4)E\left(\frac{1}{1+s^2}\right) - 2s^2K\left(\frac{1}{1+s^2}\right)}{2\sqrt{1+s^2}}\right)^2 - 4(1+2s^2) < 0. \tag{V.74}$$

$$\begin{aligned}\frac{\partial \psi_2\left(\alpha, \gamma\bar{\sigma}^2; \gamma^{-1}\right)}{\partial\gamma} &= 4(1+2\gamma\bar{s}^2)\alpha^2 - 4\alpha\left(\int_0^{\frac{\pi}{2}} \frac{2\sin^2\theta+\gamma\bar{s}^2}{(\sin^2\theta+\gamma\bar{s}^2)^{\frac{1}{2}}}\mathrm{d}\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\gamma^2\bar{s}^4}{(\sin^2\theta+\gamma\bar{s}^2)^{\frac{3}{2}}}\mathrm{d}\theta\right)+4\\ &\overset{(a)}{=} (1+2s^2)\alpha^2 - 4\alpha\left(\int_0^{\frac{\pi}{2}} \frac{2\sin^2\theta+s^2}{\left(\sin^2\theta+s^2\right)^{\frac{1}{2}}}\mathrm{d}\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{s^4}{(\sin^2\theta+s^2)^{\frac{3}{2}}}\mathrm{d}\theta\right)+4\\ &\overset{(b)}{=} 4(1+2s^2)\alpha^2 - 4\alpha\left(\frac{(5s^2+4)E\left(\frac{1}{1+s^2}\right) - 2s^2K\left(\frac{1}{1+s^2}\right)}{2\sqrt{1+s^2}}\right)+4, \end{aligned}\tag{V.73}$$

Further, to prove (V.74), it is sufficient to prove that the following two inequalities hold:

$$(5s^2+4)E\left(\frac{1}{1+s^2}\right) - 2s^2K\left(\frac{1}{1+s^2}\right) > 0, \tag{V.75a}$$

and

$$(5s^2+4)E\left(\frac{1}{1+s^2}\right) - 2s^2K\left(\frac{1}{1+s^2}\right) < 4\sqrt{1+s^2}\sqrt{1+2s^2}. \tag{V.75b}$$

We first prove (V.75a). It is sufficient to prove the following

$$(4s^2+4)E\left(\frac{1}{1+s^2}\right) - 2s^2K\left(\frac{1}{1+s^2}\right) > 0. \tag{V.76}$$

Applying a variable change $x=\frac{1}{1+s^2}$, we can rewrite (V.76) as

$$\frac{4E(x)-2(1-x)K(x)}{x} > 0.$$

The above inequality holds since

$$\begin{aligned}4E(x)-2(1-x)K(x) &> 2E(x)-2(1-x)K(x)\\ &= 2T(x) > 0,\end{aligned}$$

where the last equality is from the definition of $T(x)$ in (A.1).

We next prove (V.75b). Again, applying the variable change $x=\frac{1}{1+s^2}$ and after some straightforward manipulations, we can rewrite (V.75b) as

$$h(x)/x < 0, \quad x\in(0,1),$$

where

$$h(x) \overset{\Delta}{=} (5-x)E(x) - 2(1-x)K(x) - 4\sqrt{2-x} < 0.$$

Hence, we only need to prove $h(x)<0$ for $0<x<1$. First, we note that $\lim_{x\to1^-} h(x)=0$, from the fact that $E(1)=1$ and $\lim_{x\to1^-}(1-x)K(x)=0$ (see Lemma 18 (i)). We finish the proof by showing that $h(x)$ is strictly increasing in $x\in(0,1)$. Using the identities in (A.2), we can obtain

$$h'(x) = \frac{3}{2}\frac{(1-x)(E(x)-K(x))}{x} + \frac{2}{\sqrt{2-x}}.$$

To prove $h'(x)>0$, it is equivalent to prove

$$\begin{aligned}&\frac{4x}{3(1-x)\sqrt{2-x}}\\ &\quad > K(x)-E(x)\\ &\quad = \int_0^{\frac{\pi}{2}} \frac{1}{(1-x\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta - \int_0^{\frac{\pi}{2}}(1-x\sin^2\theta)^{\frac{1}{2}}\mathrm{d}\theta\\ &\quad = \int_0^{\frac{\pi}{2}} \frac{x\sin^2\theta}{(1-x\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta. \end{aligned}\tag{V.77}$$

Noting $0<x<1$, we can get the following

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{x\sin^2\theta}{(1-x\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta &< \int_0^{\frac{\pi}{2}} \frac{x\sin^2\theta}{1-x\sin^2\theta}\mathrm{d}\theta\\ &= \frac{\pi}{2}\left(\frac{1}{\sqrt{1-x}}-1\right).\end{aligned}$$

Hence, to prove (V.77), it suffices to prove

$$\frac{4x}{3(1-x)\sqrt{2-x}} > \frac{\pi}{2}\left(\frac{1}{\sqrt{1-x}}-1\right),$$

which can be reformulated as

$$\frac{8}{3\pi}\frac{1}{\sqrt{2-x}} > \frac{\sqrt{1-x}}{1+\sqrt{1-x}}.$$

The inequality holds since

$$\frac{8}{3\pi}\frac{1}{\sqrt{2-x}} > \frac{8}{3\pi}\frac{1}{\sqrt{2}} > \frac{1}{2}, \quad \forall x\in(0,1),$$

and

$$\frac{\sqrt{1-x}}{1+\sqrt{1-x}} < \frac{1}{2}, \quad \forall x\in(0,1).$$

□

- **Main proof**

We now return to the main proof for Lemma 5. Notice that by Lemma 10, $(\alpha_{t_0}, \sigma^2_{t_0})$ cannot fall below the curve $L(\alpha;\delta)$ for $t_0\geq1$. Hence, for $\mathcal{R}_2$, we can focus on the region above $L(\alpha;\delta)$ (including $L(\alpha;\delta)$), which we denote as $\mathcal{R}_{2a}$. See Fig. 7 for illustration.

We will first prove that if $(\alpha,\sigma^2)\in\mathcal{R}_1\cup\mathcal{R}_{2a}$, then the next iterates $\psi_1(\alpha,\sigma^2)$ and $\psi_2(\alpha,\sigma^2)$ satisfy the following:

$$\psi_1(\alpha,\sigma^2) \geq B_1(\alpha,\sigma^2), \tag{V.78a}$$

and

$$\psi_2(\alpha,\sigma^2) \leq B_2(\alpha,\sigma^2), \tag{V.78b}$$

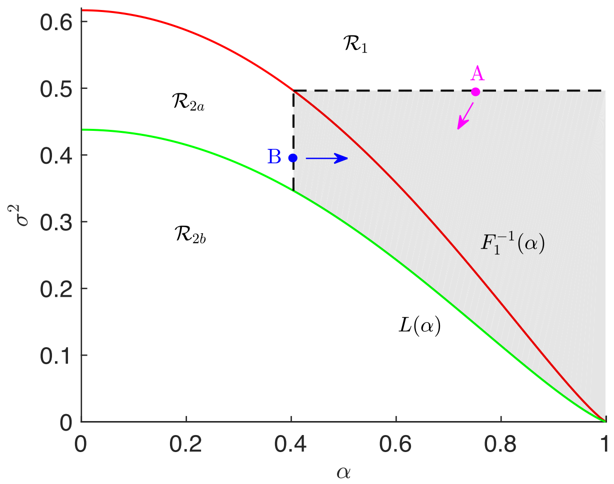

Fig. 7. Illustration of the convergence behavior. $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined in Definition 4. For both point A and point B, $B_1(\alpha,\sigma^2)$ and $B_2(\alpha,\sigma^2)$ are given by the two dashed lines. After one iteration, $\mathcal{R}_{2b}$ will not be achievable and we can focus on $\mathcal{R}_{2a}$.

where $B_1(\alpha,\sigma^2)$ and $B_2(\alpha,\sigma^2)$ are defined as

$$B_1(\alpha,\sigma^2) \triangleq \min\left\{\alpha, F_1(\sigma^2)\right\},$$
$$B_2(\alpha,\sigma^2) \triangleq \max\left\{\sigma^2, F_1^{-1}(\alpha)\right\}. \quad \text{(V.79)}$$

Note that when (α,σ^2) is on F_1^{-1} (i.e., $\sigma^2 = F_1^{-1}(\alpha)$), equalities in (V.78a) and (V.78b) can be achieved. Further, this is the only case when either of the equality is achieved. Also, it is easy to see that if (α,σ^2) is on F_1^{-1}, then $(\psi_1(\alpha,\sigma^2), \psi_2(\alpha,\sigma^2))$ cannot be on F_1^{-1}.
Since F_1^{-1} separates $\mathcal{R}_1$ and $\mathcal{R}_{2a}$, (V.79) can also be written as

$$\left[B_1(\alpha,\sigma^2), B_2(\alpha,\sigma^2)\right] = \begin{cases}[F_1(\sigma^2), \sigma^2] & \text{if } (\alpha,\sigma^2)\in\mathcal{R}_1,\\ [\alpha, F_1^{-1}(\alpha)] & \text{if } (\alpha,\sigma^2)\in\mathcal{R}_{2a}.\end{cases} \quad \text{(V.80)}$$

As a concrete example, consider the situation shown in Fig. 7. In this case, for both point A and point B, $B_1(\alpha,\sigma^2)$ and $B_2(\alpha,\sigma^2)$ are given by the two dashed lines. This directly follows from (V.80) by noting that point A is in region $\mathcal{R}_1$ and point B is in region $\mathcal{R}_{2a}$. Let $\mathcal{R}_{2a}\backslash F_1^{-1}(\alpha)$ be a shorhand for $\{(\alpha,\sigma^2)|(\alpha,\sigma^2)\in\mathcal{R}_{2a}, \alpha \neq F_1(\sigma^2)\}$. To prove the strict inequality in (V.78), we deal with $(\alpha,\sigma^2)\in\mathcal{R}_1$ and $(\alpha,\sigma^2)\in\mathcal{R}_{2a}\backslash F_1^{-1}(\alpha)$ separately.

1) Assume that $(\alpha,\sigma^2)\in\mathcal{R}_1$. Using (V.80), the inequality in (V.78) can be rewritten as

$$\psi_1(\alpha,\sigma^2) > F_1(\sigma^2) \quad \text{and} \quad \psi_2(\alpha,\sigma^2) < \sigma^2. \quad \text{(V.81)}$$

Since $(\alpha,\sigma^2)\in\mathcal{R}_1$, we have $\sigma^2 > F_1^{-1}(\alpha)$. Then, applying (V.4) proves $\psi_1(\alpha,\sigma^2) > F_1(\sigma^2)$. Further, using Lemma 3, we have $\sigma^2 > F_1^{-1}(\alpha) > F_2(\alpha)$. Also, Lemma 4 guarantees that $\sigma^2 < \sigma^2_{\max}$. Hence, $F_1^{-1}(\alpha) < \sigma^2 < \sigma^2_{\max}$ and applying Lemma 8 (iv) yields $\psi_2(\alpha,\sigma^2) < \sigma^2$.

2) We now consider the case where $(\alpha,\sigma^2)\in\mathcal{R}_{2a}\backslash F_1^{-1}(\alpha)$. Similar to (V.81), we need to prove

$$\psi_1(\alpha,\sigma^2) > \alpha \quad \text{and} \quad \psi_2(\alpha,\sigma^2) < F_1^{-1}(\alpha). \quad \text{(V.82)}$$

The inequality $\psi_1(\alpha,\sigma^2) > \alpha$ can be proved by the global attractiveness in Lemma 7 (iii) and the fact that $\sigma^2 < F_1^{-1}(\alpha)$ when $(\alpha,\sigma^2)\in\mathcal{R}_{2a}\backslash F_1^{-1}(\alpha)$. The proof for $\psi_2(\alpha,\sigma^2) < F_1^{-1}(\alpha)$ is considerably more complicated and is detailed in Lemma 16 below.

Lemma 16: For any $(\alpha,\sigma^2)\in\mathcal{R}_{2a}$ (see Definition 4) and $\delta \geq \delta_{\text{AMP}}$, the following holds:

$$\psi_2(\alpha,\sigma^2;\delta) < F_1^{-1}(\alpha), \quad \text{(V.83)}$$

where ψ_2 is the SE map in (II.2c) *and F_1^{-1} is the inverse of F_1 defined in Lemma 7.*

Proof:
The following holds when $(\alpha,\sigma^2)\in\mathcal{R}_{2a}$:

$$\psi_2(\alpha,\sigma^2;\delta) \leq \max_{\hat{\sigma}^2\in\mathcal{D}_\alpha} \psi_2(\alpha,\hat{\sigma}^2;\delta),$$

where

$$\mathcal{D}_\alpha \triangleq \left\{\hat{\sigma}^2 \middle| L(\alpha;\delta) \leq \sigma^2 \leq F_1^{-1}(\alpha)\right\}. \quad \text{(V.84)}$$

Hence, to prove (V.83), it suffices to prove that the following holds for any $\delta \geq \delta_{\text{AMP}}$ and $\alpha\in[0,1]$:

$$\max_{\hat{\sigma}^2\in\mathcal{D}_\alpha} \psi_2(\alpha,\hat{\sigma}^2;\delta) < F_1^{-1}(\alpha). \quad \text{(V.85)}$$

We next prove (V.85). We consider the three different cases:

(i) $\alpha\in[\alpha_*,1]$ and all $\delta\in[\delta_{\text{AMP}},\infty)$, where α_* is defined in (V.9).
(ii) $\alpha\in[0,\alpha_*)$ and $\delta\in[\delta_{\text{AMP}},17]$.
(iii) $\alpha\in[0,\alpha_*)$ and $\delta\in(17,\infty)$.

Case (i): Lemma 8 (v) shows that ψ_2 is an increasing function of σ^2 in $\mathbb{R}_+$. Hence, by noting (V.84), we have

$$\max_{\hat{\sigma}^2\in\mathcal{D}_\alpha} \psi_2(\alpha,\hat{\sigma}^2;\delta) = \psi_2(\alpha, F_1^{-1}(\alpha);\delta).$$

Therefore, proving (V.89) reduces to proving

$$\psi_2(\alpha, F_1^{-1}(\alpha);\delta) \leq F_1^{-1}(\alpha). \quad \text{(V.86)}$$

Finally, (V.86) follows from the global attractiveness property in Lemma 8 (iv) and the inequality $F_1^{-1}(\alpha) > F_2(\alpha;\delta)$ in Lemma 3.
Case (ii): We will prove that the following holds for $\alpha\in[0,\alpha_*)$ and $\delta\in[\delta_{\text{AMP}},17]$ (at the end of this proof)

$$\max_{\hat{\sigma}^2\in\mathcal{D}_\alpha} \psi_2(\alpha,\sigma^2;\delta) = \max\left\{\psi_2(\alpha, L(\alpha;\delta);\delta),\ \psi_2(\alpha, F_1^{-1}(\alpha);\delta)\right\}. \quad \text{(V.87)}$$

Namely, the maximum of ψ_2 over σ^2 is achieved at either $\sigma^2 = L(\alpha;\delta)$ or $\sigma^2 = F_1^{-1}(\alpha)$. Hence, we only need to prove that the following holds for any $\alpha \in [0,\alpha_*)$ and $\delta \geq \delta_{\mathrm{AMP}}$:

$$\max\left\{\psi_2(\alpha, L(\alpha;\delta);\delta),\ \ \psi_2(\alpha, F_1^{-1}(\alpha);\delta)\right\} \leq F_1^{-1}(\alpha). \quad \text{(V.88)}$$

In the sequel, we first use (V.87) to prove (V.85), and the proof for (V.87) will come at the end of this proof.

Firstly, it is easy to see that $\psi_2(\alpha, F_1^{-1}(\alpha);\delta)$ is a decreasing function of δ, since $\psi_2(\alpha,\sigma^2;\delta)$ is a decreasing function of δ and $F_1^{-1}(\alpha)$ does not depend on δ. Further, Lemma 15 shows that $\psi_2(\alpha, L(\alpha;\delta);\delta)$ is also a decreasing function of δ. (Notice that unlike $F_1^{-1}(\alpha)$, $L(\alpha;\delta)$ depends on δ, and thus Lemma 15 is nontrivial.) Hence, to prove (V.88) for $\delta \geq \delta_{\mathrm{AMP}}$, it suffices to prove (V.88) for $\delta = \delta_{\mathrm{AMP}}$, namely,

$$\max\left\{\psi_2(\alpha, L(\alpha;\delta);\delta_{\mathrm{AMP}}), \psi_2(\alpha, F_1^{-1}(\alpha);\delta_{\mathrm{AMP}})\right\} \leq F_1^{-1}(\alpha). \quad \text{(V.89)}$$

When $\delta = \delta_{\mathrm{AMP}}$, we prove in Lemma 14 that ψ_2 is an increasing function of σ^2 in $\sigma^2 \in [L(\alpha;\delta_{\mathrm{AMP}}),\infty)$. (Such monotonicity generally does not hold if δ is too large.) Further, Lemma 12 shows that $F_1^{-1}(\alpha) > L(\alpha;\delta_{\mathrm{AMP}})$. Hence,

$$\psi_2(\alpha, L(\alpha;\delta);\delta_{\mathrm{AMP}}) \leq \psi_2(\alpha, F_1^{-1}(\alpha);\delta_{\mathrm{AMP}}),$$

and thus proving (V.89) reduces to proving

$$\psi_2(\alpha, F_1^{-1}(\alpha);\delta_{\mathrm{AMP}}) \leq F_1^{-1}(\alpha),$$

which follows from the same argument as that for (V.86).

Case (iii): Lemma 8 (iii) shows that $\psi_2(\alpha;\sigma^2;\delta) \leq \frac{4}{\delta}$ for any $\sigma^2 \in [0,\sigma^2_{\max}]$. It is easy to see that $\mathcal{D}_\alpha \subset [0,\sigma^2_{\max}]$, and thus

$$\max_{\sigma^2\in\mathcal{D}_\alpha} \psi_2(\alpha,\sigma^2;\delta) \leq \frac{4}{\delta} \leq \frac{4}{17} \approx 0.235. \quad \text{(V.90)}$$

Further, Lemma 9 shows that $F_1^{-1} : [0,1] \mapsto [0,\pi^2/16]$ is monotonically decreasing. Hence,

$$F_1^{-1}(\alpha) > F_1^{-1}(\alpha_*) \approx 0.415, \quad \text{(V.91)}$$

where the numerical constant is calculated from the closed form formula $F_1^{-1}(\alpha) = \alpha^2\cdot\left[\phi_1^{-1}(\alpha)\right]^2$ (see (V.34)) and $\alpha_* \approx 0.5274$ (from (V.9)). Comparing (V.90) and (V.91) shows that (V.85) holds in this case.

It only remains to prove (V.87). We have shown in (V.17) that

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} = \frac{4}{\delta\alpha}\Bigg(\alpha - \underbrace{\frac{1}{2\sqrt{1+s^2}}E\left(\frac{1}{1+s^2}\right)}_{f(s)}\Bigg), \quad \text{(V.92)}$$

where $s \triangleq \sigma/\alpha$. Further, we have proved in (V.19) that $f(s)$ is strictly increasing on $[0,s_*)$ and strictly decreasing on (s_*,∞), where s_* is defined in (V.24). Hence, when $f(0)=0.5<\alpha<f(s_*)=\alpha_*$, there exist two solutions to

$$\alpha = f(s),$$

denoted as $s_1(\alpha)$ and $s_2(\alpha)$, respectively. Also, from (V.92) and noting the definition $s=\sigma/\alpha$, we have

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} > 0 \Longleftrightarrow \sigma^2 \in \left[0,\sigma_1^2(\alpha)\right) \cup \left(\sigma_2^2(\alpha),\infty\right),$$

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2} \leq 0 \Longleftrightarrow \sigma^2 \in \left[\sigma_1^2(\alpha),\sigma_2^2(\alpha)\right],$$

where $\sigma_1^2(\alpha) \triangleq \alpha^2\ s_1^2(\alpha)$ and $\sigma_2^2(\alpha) \triangleq \alpha^2\ s_2^2(\alpha)$. Hence, for fixed α where $\alpha\in(f(0),f(s_*))$, $\sigma_1^2(\alpha)$ is a local maximum of ψ_2 and $\sigma_2^2(\alpha)$ is a local minimum. Clearly, if

$$L(\alpha;\delta) \geq \sigma_1^2(\alpha), \quad \text{(V.93)}$$

then the maximum of ψ_2 over $\sigma^2 \in [L(\alpha;\delta), F_1^{-1}(\alpha)]$ can only happen at either $L(\alpha;\delta)$ or $F_1^{-1}(\alpha)$, which will prove (V.87). Further, for the degenerate case $\alpha\in(0,f(0))$, ψ_2 only has a local minimum, and it is easy to see that (V.87) also holds. Thus, we only need to prove that (V.93) holds when $\delta<17$. This can be proved as follows:

$$\sigma_1^2(\alpha) \overset{(a)}{\leq} s_*^2\cdot\alpha^2 \overset{(b)}{\leq} s_*^2\cdot\alpha_*^2, \quad \text{(V.94)}$$

where (a) is from the fact that $s_1(\alpha)\leq s_*$ and (b) is from our assumption $\alpha\leq\alpha_*$. On the other hand, since $L(\alpha)$ is a decreasing function of α (see Lemma 11), and thus for $\alpha\leq\alpha_*$ we have

$$L(\alpha;\delta) \geq L(\alpha_*;\delta) = \frac{4}{\delta}\left(1 - \frac{\phi_2^2(\phi_1^{-1}(\alpha_*))}{4\left[1+(\phi_1^{-1}(\alpha_*))^2\right]}\right), \quad \text{(V.95)}$$

where the last step is from Definition V.50. Based on (V.94) and (V.95), we see that $L(\alpha;\delta)>\sigma_1^2(\alpha)$ for $\alpha\leq\alpha_*$ if

$$\delta \leq \frac{4}{s_*^2\cdot\alpha_*^2}\left(1 - \frac{\phi_2^2(\phi_1^{-1}(\alpha_*))}{4\left[1+(\phi_1^{-1}(\alpha_*))^2\right]}\right) \approx 17.04,$$

where the numerical constant is calculated based on the definition of α_* in (V.23), the definition of s_* in (V.24), and that of ϕ_1 and ϕ_2 in Definition V.50. Hence, the condition $\delta<17$ is enough for our purpose. This concludes our proof. □

Now we turn our attention to the proof of part (i) of Lemma 5. Suppose that $(\alpha,\sigma^2) \in \mathcal{R}_1\cup\mathcal{R}_{2a}$. Then, using (V.78) and based on the fact that $F_1(\alpha)$ is a strictly decreasing function, we know that

$(\psi_1(\alpha,\sigma^2),\psi_2(\alpha,\sigma^2)) \in \mathcal{R}_1 \cup \mathcal{R}_2$. (See Definition 4.) Further, Lemma 6 shows that $(\psi_1(\alpha,\sigma^2),\psi_2(\alpha,\sigma^2)) \notin \mathcal{R}_{2b}$. Hence, $(\psi_1(\alpha,\sigma^2),\psi_2(\alpha,\sigma^2)) \in \mathcal{R}_1 \cup \mathcal{R}_{2a}$. Applying this argument recursively shows that if $(\alpha_{t_0},\sigma^2_{t_0}) \in \mathcal{R}_1 \cup \mathcal{R}_{2a}$, then $(\alpha_t,\sigma_t^2) \in \mathcal{R}_1 \cup \mathcal{R}_{2a}$ for all $t > t_0$. An illustration of the situation is shown in Fig. 7.

Now we can discuss the proof of part (ii) of Lemma 5. To proceed, we introduce two auxiliary sequences $\{\tilde{\alpha}_{t+1}\}_{t\geq t_0}$ and $\{\tilde{\sigma}^2_{t+1}\}_{t\geq t_0}$, defined as:

$$\tilde{\alpha}_{t+1} = B_1(\alpha_t,\sigma_t^2) \quad \text{and} \quad \tilde{\sigma}^2_{t+1} = B_2(\alpha_t,\sigma_t^2), \tag{V.96}$$

where B_1 and B_2 are defined in (V.79). Note that the definitions of $B_1(\alpha,\sigma^2)$ and $B_2(\alpha,\sigma^2)$ require $(\alpha,\sigma^2) \in \mathcal{R}_1 \cup \mathcal{R}_{2a}$, and such requirement is satisfied here due to part (i) of this lemma. Noting the SE update $\alpha_{t+1} = \psi_1(\alpha_t,\sigma_t^2)$ and $\sigma^2_{t+1} = \psi_2(\alpha_t,\sigma_t^2)$, and recall the inequalities in (V.78), we obtain the following:

$$\alpha_{t+1} \geq \tilde{\alpha}_{t+1} \quad \text{and} \quad \sigma^2_{t+1} \leq \tilde{\sigma}^2_{t+1}, \quad \forall t \geq t_0. \tag{V.97}$$

Namely, $\{\tilde{\alpha}_{t+1}\}_{t\geq t_0}$ and $\{\tilde{\sigma}^2_{t+1}\}_{t\geq t_0}$ are "worse" than $\{\alpha_{t+1}\}_{t\geq t_0}$ and $\{\sigma^2_{t+1}\}_{t\geq t_0}$, respectively, at each iteration. We next prove that

$$\lim_{t\to\infty} \tilde{\alpha}_{t+1} = 1 \quad \text{and} \quad \lim_{t\to\infty} \tilde{\sigma}^2_{t+1} = 0, \tag{V.98}$$

which together with (V.97), and the fact that $\alpha_{t+1} \leq 1$ and $\sigma_{t+1} > 0$ (since $(\alpha_t,\sigma_t^2) \in \mathcal{R}_{2a}$), leads to the results we want to prove:

$$\lim_{t\to\infty} \alpha_{t+1} = 1 \quad \text{and} \quad \lim_{t\to\infty} \sigma^2_{t+1} = 0.$$

It remains to prove (V.98). First, notice that $\tilde{\alpha}_{t+1} \leq 1$ and $\tilde{\sigma}^2_{t+1} \geq 0$ ($\forall t \geq t_0$), from the definition in (V.79). We then show that the sequence $\{\tilde{\alpha}_{t+1}\}_{t\geq t_0}$ is monotonically non-decreasing and $\{\tilde{\sigma}^2_{t+1}\}_{t\geq t_0}$ is monotonically non-increasing, namely,

$$\tilde{\alpha}_{t+2} \geq \tilde{\alpha}_{t+1} \quad \text{and} \quad \tilde{\sigma}^2_{t+2} \leq \tilde{\sigma}^2_{t+1}, \quad \forall t \geq t_0, \tag{V.99}$$

and equalities of (V.99) hold only when the equalities in (V.78) hold. Then we can finish the proof by the fact that $\tilde{\alpha}$ and $\tilde{\sigma}^2$ will improve strictly in at most two consecutive iterations and the ratios $\frac{\tilde{\alpha}_{t+2}}{\tilde{\alpha}_t}, \frac{\tilde{\sigma}^2_{t+2}}{\tilde{\sigma}^2_t}$ are continuous functions of (α_t,σ_t^2) on $[\tilde{\alpha}_{t_0},1]\times[0,\sigma^2_{\max}]$. (This is essentially due to the fact that equalities in (V.78) can be achieved when $\sigma^2 = F_1^{-1}(\alpha)$, but this cannot happen in two consecutive iterations. See the discussions below (V.79).)

To prove (V.99), we only need to prove the following (based on the definition in (V.96))

$$B_1[\psi_1,\psi_2] \geq B_1(\alpha,\sigma^2)$$

and

$$B_2[\psi_1,\psi_2] \leq B_2(\alpha,\sigma^2), \quad \forall (\alpha,\sigma^2) \in \mathcal{R}_1 \cup \mathcal{R}_{2a},$$

where ψ_1 and ψ_2 are shorthands for $\psi_1(\alpha,\sigma^2)$ and $\psi_2(\alpha,\sigma^2;\delta)$. From (V.79), the above inequalities are equivalent to

$$\min\{\psi_1, F_1(\psi_2)\} \geq B_1(\alpha,\sigma^2), \tag{V.100}$$

and

$$\max\left\{\psi_2, F_1^{-1}(\psi_1)\right\} \leq B_2(\alpha,\sigma^2). \tag{V.101}$$

Note that (V.78) already proves the following

$$\psi_1 \geq B_1(\alpha,\sigma^2) \quad \text{and} \quad \psi_2 \leq B_2(\alpha,\sigma^2).$$

Hence, to prove (V.100) and (V.101), we only need to prove

$$F_1(\psi_2) \geq B_1(\alpha,\sigma^2) \quad \text{and} \quad F_1^{-1}(\psi_1) \leq B_2(\alpha,\sigma^2).$$

To prove $F_1(\psi_2) \geq B_1(\alpha,\sigma^2)$, we note that

$$\begin{aligned}\psi_2 &\overset{(a)}{\leq} B_2(\alpha,\sigma^2)\\ &\overset{(b)}{=} \max\left\{\sigma^2, F_1^{-1}(\alpha)\right\}\\ &\overset{(c)}{=} F_1^{-1}\left(\min\left\{F_1(\sigma^2),\alpha\right\}\right)\\ &\overset{(d)}{=} F_1^{-1}\left(B_1(\alpha,\sigma^2)\right),\end{aligned}$$

where (a) is from (V.78b), (b) is from (V.79), and (c) is due to the fact that F_1^{-1} is strictly decreasing, and (d) from (V.78). Hence, since F_1 is strictly decreasing, we have

$$F_1(\psi_2) \geq F_1\left[F_1^{-1}\left(B_1(\alpha,\sigma^2)\right)\right] = B_1(\alpha,\sigma^2).$$

Further, it is straightforward to see that if both inequalities are strict in (V.78) then

$$\min\{\psi_1, F_1(\psi_2)\} > B_1(\alpha,\sigma^2).$$

This shows that equalities of (V.99) hold only when the equalities in (V.78) hold.

The proof for $F_1^{-1}(\psi_1) \leq B_2(\alpha,\sigma^2)$ is similar and omitted.

F. Proof of Lemma 6: Behavior of the SE in $\mathcal{R}_0$

Suppose that $(\alpha,\sigma^2) \in \mathcal{R}_0$. From Definition 4, we have

$$\frac{\pi^2}{16} < \sigma^2 \leq \sigma^2_{\max}. \tag{V.102}$$

Further, F_1^{-1} is monotonically decreasing and hence (for $\delta > \delta_{\mathrm{AMP}}$)

$$\frac{\pi^2}{16} = F_1^{-1}(0) > F_1^{-1}(\alpha) \geq F_2(\alpha;\delta), \tag{V.103}$$

where the last inequality is due to Lemma 3. Combining (V.102) and (V.103) yields

$$F_2(\alpha;\delta) < \sigma^2 \leq \sigma^2_{\max}. \tag{V.104}$$

By the global attractiveness property in Lemma 8 (iv), (V.104) implies

$$\psi_2(\alpha;\sigma^2;\delta) < \sigma^2.$$

From the above analysis, we see that as long as $\frac{\pi^2}{16} < \sigma_t^2 \leq \sigma^2_{\max}$ (and also $0 < \alpha_t < 1$), σ^2_{t+1} will be strictly smaller than σ_t^2:

$$\sigma^2_{t+1} = \psi_2(\alpha_t;\sigma_t^2;\delta) < \sigma_t^2.$$

Hence, there exists a finite number $T \geq 1$ such that

$$\sigma_{T-1}^2 > \frac{\pi^2}{16} \quad \text{and } \sigma_T^2 \leq \frac{\pi^2}{16}.$$

Otherwise, σ_t^2 will converge to a $\bar{\sigma}^2$ in $\mathcal{R}_0$. This implies that $\bar{\sigma}^2$ is a fixed point of ψ_2 for certain value of $0 < \alpha \leq 1$. However, we know from part (i) of Lemma 9 and Lemma 3 that this cannot happen.

Based on a similar argument, we also have $\psi_1(\alpha; \sigma^2) < \alpha$ and so $\alpha_{t+1} < \alpha_t$ for $t \leq T-1$. Further, we can show that $\alpha_t > 0$ (i.e., $\alpha_t \neq 0$) for all $0 \leq t \leq T$. First, $\alpha_0 > 0$ follows from our assumption. Further, from (II.2a) we see that $\alpha_{t+1} > 0$ if $\alpha_t > 0$. Then, using a simple induction argument we prove that $\alpha_t > 0$ for all $0 \leq t \leq T$. Putting things together, we showed that there exists a finite number $T \geq 1$ such that

$$0 < \alpha_T \leq 1 \quad \text{and } \sigma_T^2 \leq \frac{\pi^2}{16}.$$

(Recall that we have proved in Lemma 4 that $\alpha_T \leq 1$.) From Definition 4, $(\alpha_T, \sigma_T^2) \in \mathcal{R}_1 \cup \mathcal{R}_2$.

VI. Proof of Theorem 3: Local Convergence of SE

We consider the two different cases separately: (1) $\delta > \delta_{\text{global}}$ and (2) $\delta < \delta_{\text{global}}$.

A. Case $\delta > \delta_{global}$

In this section, we will prove that when $\delta > \delta_{\text{global}}$ the state evolution converges to the fixed point $(\alpha, \sigma^2) = (1, 0)$ if initialized close enough to the fixed point. We first prove the following lemma, which shows that F_1^{-1} is larger than $F_2(\alpha; \delta)$ for α close to one.

Lemma 17: Suppose that $\delta > \delta_{\text{global}} = 2$. Then, there exists an $\epsilon > 0$ such that the following holds:

$$F_1^{-1}(\alpha) > F_2(\alpha; \delta), \quad \forall \alpha \in (1-\epsilon, 1). \tag{VI.1}$$

Proof: In Lemma 3, we proved that $F_1^{-1}(\alpha) > F_2(\alpha; \delta)$ holds for all $\alpha \in (0, 1)$ when $\delta > \delta_{\text{AMP}} \approx 2.5$. Here, we will prove that $F_1^{-1}(\alpha) > F_2(\alpha; \delta)$ holds for α close to 1 when $\delta > \delta_{\text{global}} = 2$. Similar to the manipulations given in Section V-D, the inequality (VI.1) can be re-parameterized into the following ($\forall s \in (0, \xi)$):

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(\sin^2\theta + s^2)^{\frac{1}{2}}} d\theta \cdot \int_0^{\frac{\pi}{2}} \frac{(1-\gamma s^2)\sin^2\theta + s^2}{(\sin^2\theta + s^2)^{\frac{1}{2}}} d\theta > 1, \tag{VI.2}$$

where $\gamma \triangleq 1 - \delta/4$ and $\xi = \phi_1^{-1}(\epsilon)$ (see (V.33) for the definition of ϕ_1). Again, it is more convenient to express (VI.2) using elliptic integrals (cf. (V.44))

$$\frac{E(x)T(x)}{x} - \frac{\gamma(1-x)T^2(x)}{x^2} > 1, \quad \forall x \in \left(\frac{1}{1+\xi}, 1\right), \tag{VI.3}$$

where we made a variable change $x \triangleq 1/(1+s^2)$. To this end, we can verify that

$$\lim_{x\to 1} \frac{E(x)T(x)}{x} - \frac{\gamma(1-x)T^2(x)}{x^2} = 1.$$

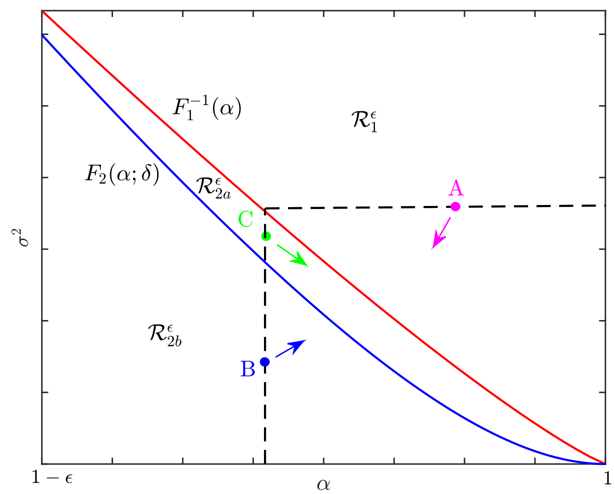

Fig. 8. Illustration of the local convergence behavior when $\delta > \delta_{\text{global}}$. For all the three points shown in the figure, B_1 and B_2 are given by the dashed lines.

To complete the proof, we only need to show that the derivative of the LHS of (VI.3) in a small neighborhood of $x = 1$ is strictly negative when $\delta > \delta_{\text{global}} = 2$. Using the formulas listed in Section VI-B, we can derive the equation shown on the top of the next page, where the last step is due to the facts that $E(x) = 1$ and $\lim_{x\to 1}(1-x)K(x) = 0$. See Section VI-B for more details. Hence, the above derivative is negative if $\gamma < \frac{1}{2}$ or $\delta > 2$ by noting the definition $\gamma = 1 - \delta/4$. □

We now turn to the proof of Lemma 3. The idea of the proof is similar to that of Theorem 2. There are some differences though, since now δ can be smaller than δ_{AMP} and some results in the proof of Theorem 2 do not hold for the case considered here. On the other hand, as we focus on the range $\alpha \in (1-\epsilon, 1) > \alpha_*$, and under this condition we know that $F_2(\sigma^2; \delta)$ is strongly globally attracting (see Lemma 8-(v)), which means that $\psi_2(\alpha, \sigma^2)$ moves towards the fixed point $F_2(\alpha; \delta)$, but cannot move to the other side of $F_2(\alpha; \delta)$.

We continue to prove the local convergence of the state evolution. We divide the region $\mathcal{R}^\epsilon \triangleq \{(\alpha, \sigma^2) | 1-\epsilon \leq \alpha \leq 1, 0 \leq \sigma^2 \leq F_1^{-1}(1-\epsilon)\}$ into the following sub-regions:

$$\begin{aligned}
\mathcal{R}_1^\epsilon &\triangleq \left\{(\alpha, \sigma^2) \middle| 1-\epsilon \leq \alpha \leq 1, F_1^{-1}(\alpha) < \sigma^2 \leq F_1^{-1}(1-\epsilon)\right\}, \\
\mathcal{R}_{2a}^\epsilon &\triangleq \left\{(\alpha, \sigma^2) \middle| 1-\epsilon \leq \alpha \leq 1, F_2(\alpha; \delta) < \sigma^2 \leq F_1^{-1}(\alpha)\right\} \\
\mathcal{R}_{2b}^\epsilon &\triangleq \left\{(\alpha, \sigma^2) \middle| 1-\epsilon \leq \alpha \leq 1, 0 \leq \sigma^2 \leq F_2(\alpha; \delta)\right\}.
\end{aligned} \tag{VI.4}$$

Similar to the proof of Lemma 5 discussed in Section V-E, we will show that if $(\alpha, \sigma^2) \in \mathcal{R}^\epsilon$ then the new states (ψ_1, ψ_2) can be bounded as follows ($\forall (\alpha, \sigma^2) \in \mathcal{R}^\epsilon$):

$$\psi_1(\alpha, \sigma^2) \geq B_1(\alpha, \sigma^2) \quad \text{and } \psi_2(\alpha, \sigma^2) \leq B_2(\alpha, \sigma^2) \tag{VI.5}$$

where

$$\begin{aligned}
B_1(\alpha, \sigma^2) &= \min\left\{\alpha, F_1(\sigma^2)\right\} \\
B_2(\alpha, \sigma^2) &= \max\left\{\sigma^2, F_1^{-1}(\alpha)\right\}.
\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E(x)T(x)}{x}-\frac{\gamma(1-x)T^2(x)}{x^2}\right)\Big|_{x\to1}\\&=\frac{2\gamma(x-4)E(x)\cdot(1-x)K(x)+[4\gamma(1-x)+x]\cdot(1-x)K^2(x)+[2\gamma(2-x)-x]E^2(x)}{2x^3}\Big|_{x\to1}\\&=\gamma-\frac{1}{2},\end{aligned}$$

Based on the strong global attractiveness of ψ_1 (Lemma 7-iii) and ψ_2 (Lemma 8-v) and the additional result (V.7), it is straightforward to show the following:

$$\begin{aligned}&\psi_1(\alpha,\sigma^2)\geq F_1(\sigma^2)\quad\text{and}\ \psi_2(\alpha,\sigma^2)\leq\sigma^2,\quad\forall(\alpha,\sigma^2)\in\mathcal{R}_1^\epsilon,\\&\psi_1(\alpha,\sigma^2)\geq\alpha\quad\text{and}\ \psi_2(\alpha,\sigma^2)\leq\sigma^2,\quad\forall(\alpha,\sigma^2)\in\mathcal{R}_{2a}^\epsilon,\\&\psi_1(\alpha,\sigma^2)\geq\alpha\quad\text{and}\ \psi_2(\alpha,\sigma^2)\leq F_2(\alpha;\delta),\quad\forall(\alpha,\sigma^2)\in\mathcal{R}_{2b}^\epsilon,\end{aligned}$$

which, together with the definitions given in (VI.4) and the fact that $F_2(\alpha;\delta)<F_1^{-1}(\alpha)$ (cf. Lemma 17), proves (VI.5). The rest of the proof follows that in Section V-E. Namely, we construct two auxiliary sequences $\{\tilde{\alpha}_{t+1}\}$ and $\{\tilde{\sigma}_{t+1}^2\}$ where

$$\tilde{\alpha}_{t+1}=B_1(\alpha_t,\sigma_t^2)\quad\text{and}\ \tilde{\sigma}_{t+1}^2=B_2(\alpha_t,\sigma_t^2),$$

and show that $\{\tilde{\alpha}_{t+1}\}$ and $\{\tilde{\sigma}_{t+1}^2\}$ monotonically converge to 1 and 0 respectively. The detailed arguments can be found in Section V-E and will not be repeated here.

B. Case $\delta<\delta_{global}$

We proved in (V.17) that

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}=\frac{4}{\delta\alpha}\Bigg(\alpha-\underbrace{\frac{1}{2\sqrt{1+s^2}}E\left(\frac{1}{1+s^2}\right)}_{f(s)}\Bigg),$$

where $s=\frac{\sigma}{\alpha}$. Hence, we have (note that $E(1)=1$)

$$\partial_2\psi_2(\alpha,0)\triangleq\frac{\partial\psi_2(\alpha,\sigma^2)}{\partial\sigma^2}\Big|_{\sigma^2=0}=\frac{4}{\delta}\left(1-\frac{1}{2\alpha}\right),\quad\forall\alpha>0.\tag{VI.6}$$

Therefore,

$$\partial_2\psi_2(\alpha,0)>1,\quad\forall\alpha>\frac{2}{4-\delta}.$$

When $\delta<\delta_{\text{global}}=2$, we have $\frac{2}{4-\delta}<1$ and therefore there exists a constant α^* that satisfies the following:

$$\frac{2}{4-\delta}<\alpha^*<1,$$

which together with (VI.6) yields

$$\partial_2\psi_2(\alpha^*,0)>1.$$

Further, as discussed in the proof of Lemma 8-(i), $\partial_2\psi_2(\alpha^*,\sigma^2)$ is a continuous function of σ^2. Hence, there exists $\xi^*>0$ such that

$$\partial_2\psi_2(\alpha^*,\sigma^2)>1,\quad\forall\sigma^2\in[0,\xi^*].\tag{VI.7}$$

Further, we have shown in (V.10) that

$$\frac{\partial\psi_2(\alpha,\sigma^2;\delta)}{\partial\sigma^2}=\frac{4}{\delta}\left(1-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sigma^2}{(\alpha^2\sin^2\theta+\sigma^2)^{\frac{3}{2}}}\mathrm{d}\theta\right),$$

and it is easy to see that $\partial_2\psi_2(\alpha,\sigma^2;\delta)$ is an increasing function of $\alpha\in(0,\infty)$. Hence, together with (VI.7) we get the following

$$\partial_2\psi_2(\alpha,\sigma^2;\delta)>1,\quad\forall(\alpha,\sigma^2)\in[\alpha^*,1]\times[0,\xi^*],$$

which means that $\psi_2(\alpha,\sigma^2)-\sigma^2$ is a strictly increasing function of σ^2 for $(\alpha,\sigma^2)\in[\alpha^*,1]\times[0,\xi^*]$. Hence,

$$\psi_2(\alpha,\sigma^2)-\sigma^2>\psi_2(\alpha,0)=\frac{4}{\delta}(1-\alpha)^2\geq0,$$

for any $(\alpha,\sigma^2)\in[\alpha^*,1]\times[0,\xi^*]$. This implies that σ^2 moves away from 0 in a neighborhood of the fixed point $(1,0)$.

Appendix A
Background on Elliptic Integrals

The functions that we have in (II.1) are related to the first and second kinds of elliptic integrals. Below we review some of the properties of these functions that will be used throughout our paper. Elliptic integrals (elliptic integral of the second kind) were originally proposed for the study of the arc length of ellipsoids. Since their appearance, elliptic integrals have appeared in many problems in physics and chemistry, such as characterization of planetary orbits. Three types of elliptic integrals are of particular importance, since a large class of elliptic integrals can be reduced to these three. We introduce two of them that are of particular interest in our work.

Definition 6: The first and second kinds of complete elliptic integrals, denoted by $K(m)$ and $E(m)$ (for $-\infty<m<1$) respectively, are defined as [61]

$$K(m)=\int_0^{\frac{\pi}{2}}\frac{1}{(1-m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta,\tag{A.1a}$$

$$E(m)=\int_0^{\frac{\pi}{2}}(1-m\sin^2\theta)^{\frac{1}{2}}\mathrm{d}\theta.\tag{A.1b}$$

For convenience, we also introduce the following definition:

$$T(m)=E(m)-(1-m)K(m).\tag{A.1c}$$

In the above definitions, we continued to use m, to follow the convention in the literature of elliptic integrals. Previously, m was defined to be the number of measurements, but such abuse of notation should not cause confusion as the exact meaning of m is usually clear from the context.

Below, we list some properties of elliptic integrals that will be used in this paper. The proofs of these properties can be found in standard references for elliptic integrals and thus omitted (e.g., [61]).

Lemma 18: The following hold for $K(m)$ and $E(m)$ defined in (A.1)*:*

(i) $K(0)=E(0)=\frac{\pi}{2}$. *Further, for $\epsilon\to 0$, $E(1-\epsilon)$ and $K(1-\epsilon)$ behave as*

$$E(1-\epsilon)=1+\frac{\epsilon}{2}\left(\log\frac{4}{\sqrt{\epsilon}}-0.5\right)+O(\epsilon^2\log(1/\epsilon))$$
$$K(1-\epsilon)=\log\left(\frac{4}{\sqrt{\epsilon}}\right)+O(\epsilon\log(1/\epsilon)).$$

(ii) *On $m\in(0,1)$, $K(m)$ is strictly increasing, $E(m)$ is strictly decreasing, and $T(m)$ is strictly increasing.*

(iii) *For $m>-1$,*

$$K(-m)=\frac{1}{\sqrt{1+m}}K\left(\frac{m}{1+m}\right),$$
$$E(-m)=\sqrt{1+m}\,E\left(\frac{m}{1+m}\right).$$

(iv) *The derivatives of $K(m)$, $E(m)$ and $T(m)$ are given by (for $m<1$)*

$$K'(m)=\frac{E(m)-(1-m)K(m)}{2m(1-m)},$$
$$E'(m)=\frac{E(m)-K(m)}{2m},$$
$$T'(m)=\frac{1}{2}K(m). \tag{A.2}$$

Furthermore, we will use a few more elliptic integrals in our work. Next lemma and its proof connects these elliptic integrals to Type I and Type II elliptic integrals.

Lemma 19: The following equalities hold for any $m\geq 0$:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{\left(1+m\sin^2\theta\right)^{\frac{3}{2}}}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\left(1+m\sin^2\theta\right)^{\frac{1}{2}}}\mathrm{d}\theta, \tag{A.3a}$$

$$\int_0^{\frac{\pi}{2}}\frac{3m\cos^2\theta}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta$$
$$=\int_0^{\frac{\pi}{2}}\frac{1+2m\sin^2\theta}{\left(1+m\sin^2\theta\right)^{\frac{1}{2}}}\mathrm{d}\theta. \tag{A.3b}$$

Proof: We will only prove (A.3b). (A.3a) can be proved in the same way. The idea is to express the integrals using elliptic integrals defined in (A.1), and then apply known properties of elliptic integrals (Lemma 18) to simplify the results. The same tricks in proving (A.3b) are used to derive other related integrals in this paper. Below, we will provide the full details for the proof of (A.3b), and will not repeat such calculations elsewhere. The LHS of (A.3b) can be rewritten as:

$$\int_0^{\frac{\pi}{2}}\frac{3m}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta-\int_0^{\frac{\pi}{2}}\frac{3m\sin^2\theta}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta$$
$$+\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta=\int_0^{\frac{\pi}{2}}\frac{1+2m\sin^2\theta}{\left(1+m\sin^2\theta\right)^{\frac{1}{2}}}\mathrm{d}\theta. \tag{A.4}$$

The equality in (A.4) can be proved by combining the identities shown at the top of the next page together with straightforward manipulations. In (A.5), as shown at the top of the next page, $K(m)$ and $E(m)$ denote the complete elliptic integrals of the first and second kinds (see (A.1)). First, consider the identity (i) in (A.5):

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta\\&=\frac{1}{m}\int_0^{\frac{\pi}{2}}(1+m\sin^2\theta)^{\frac{1}{2}}\mathrm{d}\theta-\frac{1}{m}\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta\\&\overset{(a)}{=}\frac{1}{m}\left[E(-m)-K(-m)\right]\\&\overset{(b)}{=}\frac{1}{m}\left[\sqrt{1+m}\,E\left(\frac{m}{1+m}\right)-\frac{1}{\sqrt{1+m}}K\left(\frac{m}{1+m}\right)\right],\end{aligned} \tag{A.6}$$

where (a) is from the definition of $K(m)$ and $E(m)$ in (A.1), and (b) is from Lemma 18 (iii).

Identity (ii) can be proved as follows:

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta&=-2\frac{\mathrm{d}}{\mathrm{d}m}\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta\\&=-2\frac{\mathrm{d}}{\mathrm{d}m}K(-m)\\&\overset{(a)}{=}\frac{(1+m)K(-m)-E(-m)}{m(1+m)}\\&\overset{(b)}{=}\frac{K\left(\frac{m}{1+m}\right)-E\left(\frac{m}{1+m}\right)}{m\sqrt{1+m}},\end{aligned} \tag{A.7}$$

where (a) is due to Lemma 18 (iv) and (b) is from Lemma 18 (iii).

For identity (iii), we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta\\&=\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta-m\cdot\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta\\&\overset{(a)}{=}K(-m)-m\cdot\frac{(1+m)K(-m)-E(-m)}{m(1+m)}\\&=\frac{E(-m)}{1+m}\\&\overset{(b)}{=}\frac{1}{\sqrt{1+m}}E\left(\frac{m}{1+m}\right),\end{aligned} \tag{A.8}$$

where step (a) follows from the third step of (A.7), and step (b) follows from Lemma 18 (iii).

Identity (iv) can be proved in a similar way:

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta\\&=-\frac{2}{3}\cdot\frac{\mathrm{d}}{\mathrm{d}m}\int_0^{\frac{\pi}{2}}\frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta\\&\overset{(a)}{=}-\frac{2}{3}\cdot\frac{\mathrm{d}}{\mathrm{d}m}\frac{E(-m)}{1+m}\\&\overset{(b)}{=}\frac{(1+m)K(-m)-(1-m)E(-m)}{3m(1+m)^2}\\&\overset{(c)}{=}\frac{-(1-m)E\left(\frac{m}{1+m}\right)-K\left(\frac{m}{1+m}\right)}{3m(1+m)^{\frac{3}{2}}},\end{aligned}$$

$$\text{(i):}\quad \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{1}{2}}}\mathrm{d}\theta = \frac{(m+1)E\left(\frac{m}{1+m}\right) - K\left(\frac{m}{1+m}\right)}{m\sqrt{1+m}}, \tag{A.5a}$$

$$\text{(ii):}\quad \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta = \frac{K\left(\frac{m}{1+m}\right) - E\left(\frac{m}{1+m}\right)}{m\sqrt{1+m}}, \tag{A.5b}$$

$$\text{(iii):}\quad \int_0^{\frac{\pi}{2}} \frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta = \frac{1}{\sqrt{1+m}}E\left(\frac{m}{1+m}\right), \tag{A.5c}$$

$$\text{(iv):}\quad \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta = \frac{-(1-m)E\left(\frac{m}{1+m}\right) + K\left(\frac{m}{1+m}\right)}{3m(1+m)^{\frac{3}{2}}}, \tag{A.5d}$$

$$\text{(v):}\quad \int_0^{\frac{\pi}{2}} \frac{1}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta = \frac{2(m+2)E\left(\frac{m}{1+m}\right) - K\left(\frac{m}{1+m}\right)}{3(1+m)^{\frac{3}{2}}}, \tag{A.5e}$$

where (a) is from the third step of (A.8), step (b) is from Lemma 18 (iv) and (c) is from Lemma 18 (iii).

Lastly, identity (v) can be proved as follows:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{(1+m\sin^2\theta)^{\frac{3}{2}}}\mathrm{d}\theta - m\cdot\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1+m\sin^2\theta)^{\frac{5}{2}}}\mathrm{d}\theta \\
&\overset{(a)}{=} \frac{E(-m)}{1+m} - m\cdot\frac{(1+m)K(-m)-(1-m)E(-m)}{3m(1+m)^2} \\
&\overset{(b)}{=} \frac{2(m+2)E\left(\frac{m}{1+m}\right) - K\left(\frac{m}{1+m}\right)}{3(1+m)^{\frac{3}{2}}},
\end{aligned}$$

where step (a) follows from the derivations of the previous two identities and (b) is again due to Lemma 18 (iii). □

REFERENCES

[1] E. J. Candès, T. Strohmer, and V. Voroninski, "PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming," *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, Aug. 2013.

[2] P. Netrapalli, P. Jain, and S. Sanghavi, "Phase retrieval using alternating minimization," in *Proc. Adv. Neural Inf. Process. Syst.*, 2013, pp. 2796–2804.

[3] Y. C. Eldar and S. Mendelson, "Phase retrieval: Stability and recovery guarantees," *Appl. Comput. Harmon. Anal.*, vol. 36, no. 3, pp. 473–494, May 2014.

[4] E. J. Candès, X. Li, and M. Soltanolkotabi, "Phase retrieval via wirtinger flow: Theory and algorithms," *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1985–2007, Apr. 2015.

[5] Y. Chen and E. J. Candès, "Solving random quadratic systems of equations is nearly as easy as solving linear systems," *Commun. Pure Appl. Math.*, vol. 70, pp. 822–883, May 2017.

[6] G. Wang, G. B. Giannakis, and Y. C. Eldar, "Solving systems of random quadratic equations via truncated amplitude flow," *IEEE Trans. Inf. Theory*, vol. 64, no. 2, pp. 773–794, Feb. 2018.

[7] H. Zhang and Y. Liang, "Reshaped wirtinger flow for solving quadratic system of equations," in *Proc. Adv. Neural Inf. Process. Syst.*, 2016, pp. 2622–2630.

[8] T. Goldstein and C. Studer, "PhaseMax: Convex phase retrieval via basis pursuit," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2675–2689, Apr. 2018.

[9] S. Bahmani and J. Romberg, "Phase retrieval meets statistical learning theory: A flexible convex relaxation," in *Proc. 20th Int. Conf. Artif. Intell. Statist. (AISTATS)*, Fort Lauderdale, FL, USA, vol. 54, Apr. 2017, pp. 252–260.

[10] T. Cai, X. Li, and Z. Ma, "Optimal rates of convergence for noisy sparse phase retrieval via thresholded wirtinger flow," *Ann. Statist.*, vol. 44, no. 5, pp. 2221–2251, 2016.

[11] J. Sun, Q. Qu, and J. Wright, "A geometric analysis of phase retrieval," *Found. Comput. Math.*, vol. 18, pp. 1131–1198, pp. 1–68, Oct. 2018.

[12] M. Soltanolkotabi. (2017). "Structured signal recovery from quadratic measurements: Breaking sample complexity barriers via nonconvex optimization." [Online]. Available: https://arxiv.org/abs/1702.06175

[13] J. C. Duchi and F. Ruan. (2017). "Solving (most) of a set of quadratic equalities: Composite optimization for robust phase retrieval." [Online]. Available: https://arxiv.org/abs/1705.02356

[14] Y. M. Lu and G. Li, "Spectral initialization for nonconvex estimation: Highdimensional limit and phase transitions," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Aachen, Germany, 2017, pp. 3015–3019.

[15] D. Davis, D. Drusvyatskiy, and C. Paquette. (2017). "The nonsmooth landscape of phase retrieval." [Online]. Available: https://arxiv.org/abs/1711.03247

[16] Y. S. Tan and R. Vershynin. (2017). "Phase retrieval via randomized Kaczmarz: Theoretical guarantees." [Online]. Available: https://arxiv.org/abs/1706.09993

[17] H. Jeong and C. S. Güntürk. (2017). "Convergence of the randomized Kaczmarz method for phase retrieval." [Online]. Available: https://arxiv.org/abs/1706.10291

[18] W.-J. Zeng and H. So. (2017). "Coordinate descent algorithms for phase retrieval." [Online]. Available: https://arxiv.org/abs/1706.03474

[19] M. Mondelli and A. Montanari. (2017). "Fundamental limits of weak recovery with applications to phase retrieval." [Online]. Available: https://arxiv.org/abs/1708.05932

[20] O. Dhifallah and Y. M. Lu. (2017). "Fundamental limits of PhaseMax for phase retrieval: A replica analysis." [Online]. Available: https://arxiv.org/abs/1708.03355

[21] O. Dhifallah, C. Thrampoulidis, and Y. M. Lu. (2017). "Phase retrieval via linear programming: Fundamental limits and algorithmic improvements." [Online]. Available: https://arxiv.org/abs/1710.05234

[22] E. Abbasi, F. Salehi, and B. Hassibi, "Performance of real phase retrieval," in *Proc. Int. Conf. Sampling Theory Appl. (SampTA)*, Jul. 2017, pp. 101–105.

[23] Q. Qu, Y. Zhang, Y. C. Eldar, and J. Wright. (2017). "Convolutional phase retrieval via gradient descent." [Online]. Available: https://arxiv.org/abs/1712.00716

[24] L. Zheng, A. Maleki, H. Weng, X. Wang, and T. Long, "Does ℓ_p -minimization outperform ℓ_1 -minimization?" *IEEE Trans. Inf. Theory*, vol. 63, no. 11, pp. 6896–6935, Nov. 2017.

[25] S. Rangan, "Generalized approximate message passing for estimation with random linear mixing," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jul. 2011, pp. 2168–2172.

[26] J. Ma, J. Xu, and A. Maleki. (2018). "Optimization-based AMP for phase retrieval: The impact of initialization and ℓ_2-regularization." [Online]. Available: https://arxiv.org/abs/1801.01170

[27] E. T. Hale, W. Yin, and Y. Zhang, "Fixed-point continuation for ℓ_1-minimization: Methodology and convergence," *SIAM J. Optim.*, vol. 19, no. 3, pp. 1107–1130, 2008.

[28] J. Barbier, F. Krzakala, N. Macris, L. Miolane, and L. Zdeborová. (2017). “Optimal errors and phase transitions in high-dimensional generalized linear models.” [Online]. Available: https://arxiv.org/abs/1708.03395

[29] J. Xu, J. Ma, and A. Maleki, “Phase retrieval based on spectral initialization and approximate message passing,” to be published.

[30] I. Waldspurger, A. D’Aspremont, and S. Mallat, “Phase recovery, MaxCut and complex semidefinite programming,” *Math. Programm.*, vol. 149, nos. 1–2, pp. 47–81, 2015.

[31] E. J. Candès and X. Li, “Solving quadratic equations via phaselift when there are about as many equations as unknowns,” *Found. Comput. Math.*, vol. 14, no. 5, pp. 1017–1026, Oct. 2014.

[32] C. Ma, K. Wang, Y. Chi, and Y. Chen. (2017). “Implicit regularization in nonconvex statistical estimation: Gradient descent converges linearly for phase retrieval, matrix completion and blind deconvolution.” [Online]. Available: https://arxiv.org/abs/1711.10467

[33] K. Wei, “Solving systems of phaseless equations via Kaczmarz methods: A proof of concept study,” *Inverse Problems*, vol. 31, no. 12, p. 125008, 2015.

[34] Y. Chi and Y. M. Lu, “Kaczmarz method for solving quadratic equations,” *IEEE Signal Process. Lett.*, vol. 23, no. 9, pp. 1183–1187, Sep. 2016.

[35] D. Barmherzig and J. Sun. (2017). “A local analysis of block coordinate descent for gaussian phase retrieval.” [Online]. Available: https://arxiv.org/abs/1712.02083

[36] R. Ghods, A. S. Lan, T. Goldstein, and C. Studer, “Phaselin: Linear phase retrieval,” in *Proc. 52nd Annu. Conf. Inf. Sci. Syst. (CISS)*, Mar. 2018, pp. 1–6.

[37] G. Li, Y. Gu, and Y. M. Lu, “Phase retrieval using iterative projections: dynamics in the large systems limit,” in *Proc. 53rd Annu. Allerton Conf. Commun., Control, Comput. (Allerton)*, Sep. /Oct. 2015, pp. 1114–1118.

[38] C. Thrampoulidis, E. Abbasi, and B. Hassibi. (2016). “Precise error analysis of regularized m-estimators in high-dimensions.” [Online]. Available: https://arxiv.org/abs/1601.06233

[39] C. Thrampoulidis, S. Oymak, and B. Hassibi, “Regularized linear regression: A precise analysis of the estimation error,” in *Proc. 28th Conf. Learn. Theory*, 2015, pp. 1683–1709.

[40] D. L. Donoho, A. Maleki, and A. Montanari, “Message-passing algorithms for compressed sensing,” *Proc. Nat. Acad. Sci.*, vol. 106, no. 45, pp. 18914–18919, 2009.

[41] M. Bayati and A. Montanari, “The dynamics of message passing on dense graphs, with applications to compressed sensing,” *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 764–785, Feb. 2011.

[42] A. Javanmard and A. Montanari, “State evolution for general approximate message passing algorithms, with applications to spatial coupling,” *Inf. Inference, A J. IMA*, vol. 2, no. 2, pp. 115–144, Dec. 2013.

[43] P. Schniter and S. Rangan, “Compressive phase retrieval via generalized approximate message passing,” *IEEE Trans. Signal Process.*, vol. 63, no. 4, pp. 1043–1055, Feb. 2015.

[44] A. S. Bandeira, J. Cahill, D. G. Mixon, and A. A. Nelson, “Saving phase: Injectivity and stability for phase retrieval,” *Appl. Comput. Harmon. Anal.*, vol. 37, no. 1, pp. 106–125, Jul. 2014.

[45] R. Balan, P. Casazza, and D. Edidin, “On signal reconstruction without phase,” *Appl. Comput. Harmon. Anal.*, vol. 20, no. 3, pp. 345–356, 2006.

[46] S. Jalali and A. Maleki, “From compression to compressed sensing,” *Appl. Comput. Harmon. Anal.*, vol. 40, no. 2, pp. 352–385, Mar. 2016.

[47] M. Bakhshizadeh, A. Maleki, and S. Jalali. (2017). “Compressive phase retrieval of structured signal.” [Online]. Available: https://arxiv.org/abs/1712.03278

[48] J. Ma, J. Xu, and A. Maleki, “Approximate message passing for amplitude based optimization,” in *Proc. 35th Int. Conf. Mach. Learn.*, Stockholm, Sweden, vol. 80, Jul. 2018, pp. 3371–3380.

[49] M. Bayati and A. Montanari, “The LASSO risk for Gaussian matrices,” *IEEE Trans. Inf. Theory*, vol. 58, no. 4, pp. 1997–2017, Apr. 2012.

[50] A. Mousavi, A. Maleki, and R. G. Baraniuk, “Consistent parameter estimation for LASSO and approximate message passing,” *Ann. Statist.*, vol. 46, no. 1, pp. 119–148, 2018.

[51] J. Li, J.-F. Cai, and H. Zhao. (2018). “Scalable incremental nonconvex optimization approach for phase retrieval from minimal measurements.” [Online]. Available: https://arxiv.org/abs/1807.05499

[52] E. J. Candès, X. Li, and M. Soltanolkotabi, “Phase retrieval from coded diffraction patterns,” *Appl. Comput. Harmon. Anal.*, vol. 39, no. 2, pp. 277–299, Sep. 2015.

[53] B. Cakmak, O. Winther, and B. H. Fleury, “S-AMP: Approximate message passing for general matrix ensembles,” in *Proc. IEEE Inf. Theory Workshop (ITW)*, Nov. 2014, pp. 192–196.

[54] J. Ma, X. Yuan, and L. Ping, “Turbo compressed sensing with partial DFT sensing matrix,” *IEEE Signal Process. Lett.*, vol. 22, no. 2, pp. 158–161, Feb. 2015.

[55] J. Ma and L. Ping, “Orthogonal AMP,” *IEEE Access*, vol. 5, pp. 2020–2033, 2017.

[56] S. Rangan, P. Schniter, and A. K. Fletcher, “Vector approximate message passing,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2017, pp. 1588–1592.

[57] K. Takeuchi, “Rigorous dynamics of expectation-propagation-based signal recovery from unitarily invariant measurements,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2017, pp. 501–505.

[58] H. He, C.-K. Wen, and S. Jin, “Generalized expectation consistent signal recovery for nonlinear measurements,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2017, pp. 2333–2337.

[59] G. D. Anderson and M. K. Vamanamurthy, “Inequalities for elliptic integrals,” *Publications de l’Institut Math. Nouvelle Série*, vol. 37, no. 51, pp. 61–63, 1985.

[60] M.-K. Wang and Y.-M. Chu, “Asymptotical bounds for complete elliptic integrals of the second kind,” *J. Math. Anal. Appl.*, vol. 402, no. 1, pp. 119–126, Jun. 2013.

[61] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*. Berlin, Germany: Springer-Verlag, 1971.

Junjie Ma received the B.E. degree from Xidian University, China, in 2010, and his Ph.D. degree from City University of Hong Kong in 2015. He was a research fellow at the Department of Electronic Engineering, City University of Hong Kong from 2015 to 2016. Since September 2016, he has been a postdoctoral researcher at Columbia University. His current research interests are mainly on high dimensional signal processing.

Ji Xu is currently a fourth year Ph.D. student in Computer Science department at Columbia University. Previously, he holds a M.A. in Statistics from Columbia University, and a B.S. in Math with minor in Economics from Peking University. He received Kathryn and Shelby Cullom Davis International Fellowship in 2015 and Inaugural Cheung-Kong Graduate School of Business (CKGSB) Fellowship in 2018. His research interests are in algorithmic statistics, machine learning and high dimensional statistics. His works have been accepted in top tier conferences including oral presentation at NIPS 2016 and long talk presentation at ICML 2018.

Arian Maleki is an associate professor in the Department of Statistics at Columbia University. He received Ph.D. from Stanford University in 2010. Before joining Columbia University, he was a postdoctoral scholar in the department of Electrical and Computer Engineering at Rice University